Home assignments 1, Gravitation & Cosmology, 2006

To be handed in Friday, November 24

1. Consider the Levi–Civita symbol, or the epsilon symbol, defined by $\epsilon^{0123} = 1$ and total antisymmetry in the four indices (defined on a 4-dimensional space of euclidean or lorentzian signature). Show that $\epsilon^{\mu\nu\kappa\lambda}$ is a tensor density of weight -1.

   Show that the invariant integration measure is $d^4x\sqrt{|\det g|}$, *i.e.*, that

   $$\int d^4x\sqrt{|\det g|}\,\Phi(x)$$

   is independent of the choice of coordinates when Φ is a scalar. Give the integration measure for 2- and 3-dimensional euclidean space in polar and spherical coordinates.

   Show also that

   $$\tfrac{1}{24}\int d^4x\epsilon^{\mu\nu\kappa\lambda}\Psi_{\mu\nu\kappa\lambda}(x)$$

   is invariant, where Ψ is a tensor which is antisymmetric in all four indices.

2. Construct the Laplacian (in the case of euclidean signature) / the wave operator (in the case of lorentzian signature), acting on a scalar, from covariant derivatives, and give an explicit form in terms of derivatives and the affine connection. (The wave operator on Minkowski space is the operator $-\frac{\partial^2}{\partial t^2} + \nabla^2$, and the searched differential operator is the one that in a freely falling system takes that form.)

   As an alternative, one may consider the action for a massless scalar field,

   $$S = \int d^4x\sqrt{-g}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

   (for its invariance, see exercise 1), whose variation will give the wave equation. Show that this gives the same equation as the construction with covariant derivatives.

3. As an example of a space with constant negative curvature, take the simplest example, the two-dimensional "Poincaré disk". Its metric can be written as

$$ds^2 = \frac{dr^2 + r^2 d\phi^2}{(1 - \frac{r^2}{a^2})^2} \ ,$$

where $0 \leq r < a$ and $0 \leq \phi < 2\pi$ (ϕ is an angle). What is the distance from a point with coordinates (r, ϕ) to the "boundary" $r = a$? Calculate the curvature tensor (note that in two dimensions, there is only one independent component, given the properties on p. 141, so it is enough to calculate R_{1212}), the Ricci tensor and the curvature scalar. Compare to the corresponding results for a two-dimensional sphere.

Consider an infinitesimal coordinate transformation with

$$r' = r + \epsilon \frac{a^2 - r^2}{a} \cos\phi \ ,$$
$$\phi' = \phi - \epsilon \frac{a^2 + r^2}{ar} \sin\phi \ .$$

What is the metric in the new coordinates? Do you have any interpretation of the result?