Föreläsning 29/10-13

kap 1.1, 1.2, 1.3

Den elektromagnetiska modellen
(klassisk fysik)

Elektromagnetisk fältteori – studera elektriska laddningar i vila och rörelse. Vi utgår från den fundamentala storheten, elektrisk laddning Q.

Den elektriska laddningen är bevarad (i vår teori)

Vi studerar kraftverkan över avstånd $\Longrightarrow$ fältmodell.

Makroskopisk modell – makroskopiska laddningstätheter.

$$\rho(\mathbb{r},t) = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V}$$

ΔV innehåller laddning Δq, ρ är homogen i ΔV.

$[\rho] = As/m^3 = C/m^3$

$$\rho_s = \lim_{\Delta S \to 0} \frac{\Delta q}{\Delta S}$$ ytladdningstäthet

$$\rho_\ell = \lim_{\Delta \ell \to 0} \frac{\Delta q}{\Delta \ell}$$ linjeladdningstäthet

Elektrostatik 3.1, 3.2

Laddningar i vila ger upphov till krafter på andra laddningar. Vi beskriver detta mha det elektriska fältet $\mathbb{E}$.

Kraft på en testladdning: $\mathbb{F} = q\mathbb{E}$, $[\mathbb{E}] = V/m$

Elektrostatiken definieras av två postulat:

$\nabla \cdot \mathbb{E} = \rho/\varepsilon_0$

$\nabla \times \mathbb{E} = 0$

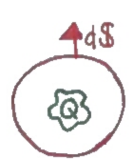
Naturkonstant: $\varepsilon_0 \approx \frac{10^{-9}}{36\pi}$ As/Vm

Gauss lag

$\nabla \cdot \mathbb{E} = \rho/\varepsilon_0$

Tag volymintegral: $\int \nabla \cdot \mathbb{E}\, dV = \int \frac{\rho}{\varepsilon_0} dV$

$$\underbrace{\oint_S \mathbb{E} \cdot d\mathbb{S} = \frac{Q}{\varepsilon_0}}_{\text{divergens-teoremet}}$$



Konservativt fält

$\nabla \times \mathbb{E} = 0$

integrera över en yta: $\int_S \nabla \times \mathbb{E} \cdot d\mathbb{S} = 0$ Stokes sats

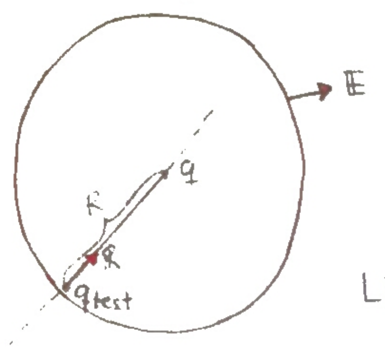
$\oint_L \mathbb{E} \cdot d\mathbb{l} = 0$

Vektoridentiteten: $\nabla \times \nabla A = 0$

→ Definiera en potential $\mathbb{E} = -\nabla V$

↑ mer om minustecknet snart...

Coulombs lag 3.3



Från symmetri: $\mathbb{E} = \hat{\mathbb{R}} E(R)$

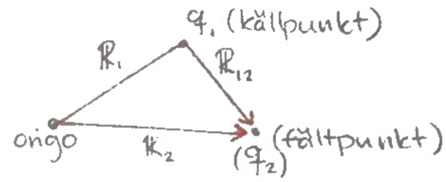
Stoppa in i Gauss lag:

$\int_S E(R)\hat{\mathbb{R}} \cdot \hat{\mathbb{R}} dS = q/\varepsilon_0$

$4\pi R^2 E(R) = q/\varepsilon_0$

Lös ut: $\mathbb{E}(R) = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} \hat{\mathbb{R}}$

Nu lade vi q i origo, men egentligen:



I kursboken
$\mathbb{R}_1 = \mathbb{R}'$
$\mathbb{R}_2 = \mathbb{R}$

Läs i boken,

$\mathbb{E}(\mathbb{R}_2) = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{R_{12}^2} \hat{\mathbb{R}}_{12}$; $|\mathbb{R}_{12}| = R_{12}$

Placera en laddning q_2 i fältet från q_1,

$\mathbb{F}_{12} = q_2 \mathbb{E}_1(\mathbb{R}_2) = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{R_{12}^2} \hat{\mathbb{R}}_{12}$ Coulombs lag

Lagen om kraft och motkraft ger $-\mathbb{F}_{12} = \mathbb{F}_{21}$

Superposition gäller ⇒ Fält från diskreta laddningar kan summeras:

$$\mathbb{E}(\mathbb{R}) = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{q_k (\mathbb{R} - \mathbb{R}_k')}{|\mathbb{R} - \mathbb{R}_k'|^3}$$

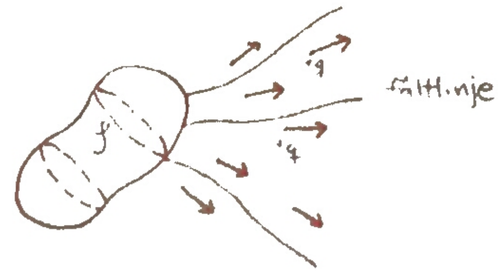
Fältet från kontinuerliga laddningsfördelningar:

$$\mathbb{E}(\mathbb{R}) = \frac{1}{4\pi\varepsilon_0} \int \frac{dQ}{R^2} \hat{\mathbb{R}} = \{\text{volym}\} = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho\, dV}{R^2} \hat{\mathbb{R}}$$

För yta $dQ = \rho_s dS$, För linje $dQ = \rho_\ell d\ell$

Studera hemma exempel: 3.4, 3.5, 3.7

Fältlinjer/Fältvektorer

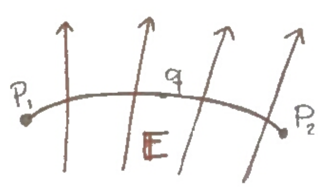


Elektrisk potential

$\nabla \times \mathbb{E} = 0$, vad ger det?

Mha $\nabla \times \nabla A \equiv 0 \Rightarrow \mathbb{E} = -\nabla V$

Med minustecknet så blir qV ett mått på den elektriska lägesenergin hos testladdning q. Fungerar som för mekanisk energi. Om man tillför arbete så ökar lägesenergin.



Fältet orsakar en kraft på q, $\mathbb{F} = q\mathbb{E}$
Om vi vill flytta q från P_1 till P_2 får vi motverka denna kraft:

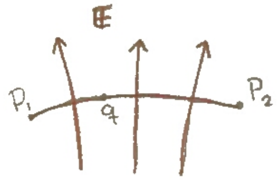
$$\mathbb{F}_{mek} = -\mathbb{F}$$

Arbete vid förflyttning:

$$W_{mek} = \int_{P_1}^{P_2} \mathbb{F}_{mek}\, d\mathbb{l} = \int_{P_2}^{P_1} \mathbb{F}\, d\mathbb{l} = q\int_{P_2}^{P_1} \mathbb{E}\, d\mathbb{l} = q\int_{P_2}^{P_1} (-\nabla V)\cdot d\mathbb{l} = q\int_{P_2}^{P_1} -dV =$$

$$= q\left[V(P_2) - V(P_1)\right]$$

Föreläsning 30/10-13



$$W_{mek} = q\int_{P_2}^{P_1} \mathbb{E}\cdot d\mathbb{l} = q\left[V(P_2) - V(P_1)\right]$$

Generalisera och lös ut: $V(R) = \int_R^{R_{ref}} \mathbb{E}\cdot d\mathbb{l} + \underbrace{V(R_{ref})}_{\text{ofta}=0}$ (startpunkt)

Beräkna potential från punktladdning,
sätt först $V(R_{ref}) = V(\infty) = 0$
Beräkna $V(R) = \int_R^{\infty} \frac{q}{4\pi\varepsilon_0}\frac{\hat{\mathbb{R}}}{r^2}\cdot\hat{\mathbb{R}}\,dr = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R}$

Generalisera:

Kom ihåg:
$\mathbb{R}_1 = \mathbb{R}'$
$\mathbb{R}_2 = \mathbb{R}$

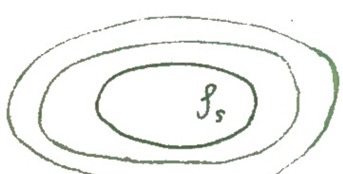
$V(\mathbb{R}_2) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1}{R_{12}}$

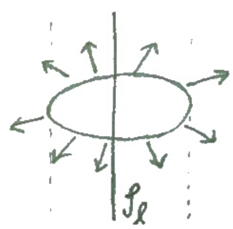
Superposition för disktreta laddningar: $V(\mathbb{R}) = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{q_k}{|\mathbb{R}-\mathbb{R}_k'|}$

För kontinuerliga laddningar: $V(\mathbb{R}) = \frac{1}{4\pi\varepsilon_0}\int\frac{dQ}{R} = \{\text{volym}\} =$

$= \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(R)}{R}\,dV'$, yta $dQ = \rho_s\,dS$, linje $dQ = \rho_\ell\,d\ell$

Studera exempel 3.9:

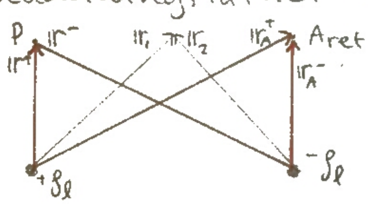




fält riktat i radiell led
cylindrisk symmetri

ekvipotentialyter

Potentialen från två "långa" parallella linjeladdningar med laddningstäthet $\pm\rho_\ell$



E-fältet från trådarna:
$\mathbb{E} = \mathbb{E}^+ + \mathbb{E}^- = \frac{\rho_\ell\hat{\mathbb{r}}_1}{2\pi\varepsilon_0 r_1} - \frac{\rho_\ell\hat{\mathbb{r}}_2}{2\pi\varepsilon_0 r_2}$

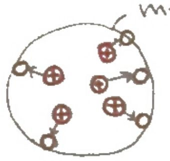
Potentialen fås som:
$V(P) - V(A) = \int_P^A \mathbb{E}\cdot d\mathbb{l} = \frac{\rho_\ell}{2\pi\varepsilon_0}\left[\int_P^A \frac{\hat{\mathbb{r}}_1}{r_1}\cdot d\mathbb{l} - \int_P^A\frac{\hat{\mathbb{r}}_2}{r_2}\cdot d\mathbb{l}\right] =$

$$= \{ \hat{r}_2 \cdot d\mathbb{l} = dr_2 ,\ \hat{r}_1 \cdot d\mathbb{l} = dr_1 \} = \frac{\rho_l}{2\pi\varepsilon_0}\left[\int_{r^+}^{r_A^+}\frac{dr_1}{r_1} - \int_{r^-}^{r_A^-}\frac{dr_2}{r_2}\right] =$$

$$= \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{r_A^+ r^-}{r^+ r_A^-}\right)$$, Låt A_{ref} ligga i $\infty \Rightarrow r_A^+ \approx r_A^- \Longrightarrow$

$$\Longrightarrow V(P) = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{r^-}{r^+}\right)$$

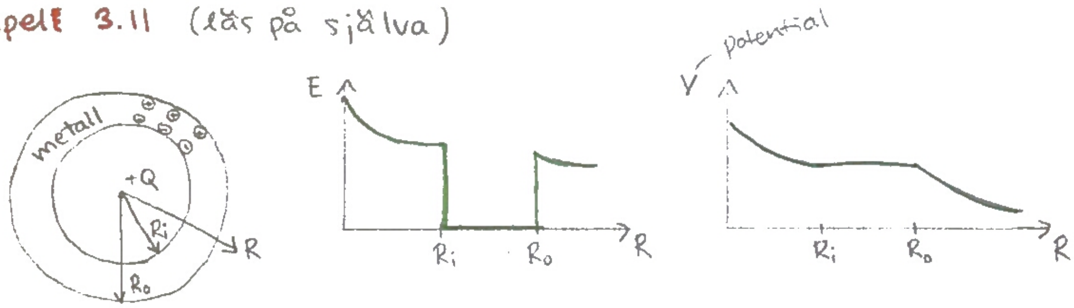
Metaller i statiskt elektriskt fält, kap 3.6



Laddningar som läggs på en metall repellerar varandra och samlas på ytan.

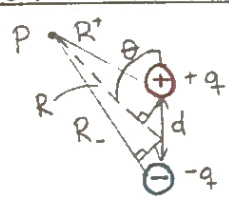
I ledaren: $\rho = 0$, $E = 0$

exempel 3.11 (läs på själva)



Föreläsning 30/10-13

Den dielektriska dipolen 3.3.1 , 3.5.1



Summera potentialbidrag: $V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$ (★)

Antag $d << R$

$$(*)\begin{cases} 1/R_+ \approx \left(R - \frac{d}{2}\cos\theta\right)^{-1} \approx \{\text{Taylor}\} \approx \frac{1}{R}\left(1 + \frac{d}{2R}\cos\theta\right) \\ 1/R_- \approx \left(R + \frac{d}{2}\cos\theta\right)^{-1} \approx \text{—}\,\text{''}\,\text{—} \approx \frac{1}{R}\left(1 - \frac{d}{2R}\cos\theta\right) \end{cases}$$

(*) och (★) $\Longrightarrow$ $V = \frac{q\,d\cos\theta}{4\pi\varepsilon_0 R^2} = \{ \mathbb{P} = q \cdot d\mathbb{l} \} = \frac{\mathbb{P}\cdot\hat{\mathbb{R}}}{4\pi\varepsilon_0 R^2}$

$\mathbb{P}$ är dipolmoment, beror av q = laddning och dl = avstånd mellan laddningarna.

E-fält: $\mathbb{E} = -\nabla V = -\hat{\mathbb{R}}\frac{\partial V}{\partial R} - \hat{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta} = \frac{p}{4\pi\varepsilon_0 R^3}\left(\hat{\mathbb{R}}\,2\cos\theta + \hat{\theta}\sin\theta\right)$

Dielektriskt material i elektrostatiskt fält 3.7

I närvaro av E-fält polariseras atomer/molekyler i ett material, dipoler bildas, molekyler roteras.

Definiera ett polarisationsfält genom att summera dipolmoment i en volym.

$$\mathbb{P} = \lim_{\Delta V \to 0}\left(\sum_{k=1}^{n\Delta V} p_k\right)/\Delta V \quad [C/m^2]$$

Potentialbidrag från materialet

Dipolmoment från dV': $dp = \mathbb{P}\cdot dV'$

Potentialbidrag: $dV = \frac{\mathbb{P}\cdot\hat{\mathbb{R}}}{4\pi\varepsilon_0 R^2}\,dV'$

Integrera en volym: $V = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\mathbb{P}\cdot\hat{\mathbb{R}}}{R^2}\,dV' = \left\{\nabla'\left(\frac{1}{R}\right) = \frac{\hat{\mathbb{R}}}{R^2}\right\} =$

$= \frac{1}{4\pi\varepsilon_0}\int_{V'}\mathbb{P}\cdot\nabla'\left(\frac{1}{R}\right)dV' = \left\{\nabla'(f\mathbb{A}) = f\nabla'\mathbb{A} + \mathbb{A}\cdot\nabla' f,\ \text{med } \mathbb{A} = \mathbb{P} \text{ och } f = 1/R\right\} =$

$= \frac{1}{4\pi\varepsilon_0}\left[\int_{V'}\nabla'\left(\frac{\mathbb{P}}{R}\right)dV' - \int_{V'}\frac{\nabla'\mathbb{P}}{R}\,dV'\right] = \{\text{divergensteorem}\} =$

$= \frac{1}{4\pi\varepsilon_0}\oint_S \underbrace{\frac{\mathbb{P}\cdot\hat{n}}{R}}_{\rho_s} dS' + \frac{1}{4\pi\varepsilon_0}\int_{V'}\underbrace{\frac{-\nabla'\mathbb{P}}{R}}_{\rho}dV'$

Identifiera: $\rho_{ps} = \mathbb{P}\cdot\hat{n}$ Polarisationsytladdningstäthet

$\rho_p = -\nabla\mathbb{P}$ Polarisationsladdningstäthet

Notera: $\underbrace{Q_p}_{\text{polarisationsladdning}} = \oint_S \rho_{ps}\,ds + \int_V \rho_p\,dv = 0$

Förskjutningsfält $\mathbb{D}$ 3.8

$$\nabla\cdot\mathbb{E} = \frac{1}{\varepsilon_0}(\rho + \rho_p) = \frac{1}{\varepsilon_0}(\rho - \nabla\cdot\mathbb{P}) \Rightarrow \nabla\cdot(\varepsilon_0\mathbb{E} + \mathbb{P}) = \rho$$

Definiera: $\mathbb{D} = \varepsilon_0\mathbb{E} + \mathbb{P}$, då blir postulatet $\nabla\mathbb{D} = \rho$

eller på integralform: $\oint_S \mathbb{D}\cdot d\mathbb{S} = Q$

fria inneslutna laddningen

Samband mellan $\mathbb{P}$ och $\mathbb{E}$

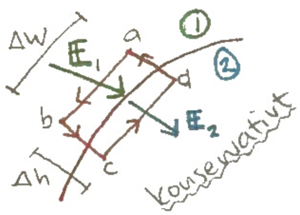
$\mathbb{P}$ beror av $\mathbb{E}$, icke-linjär tensorrelation i allmänna fallet.
Många material har ett proportionellt samband mellan $\mathbb{E}$ och $\mathbb{P}$.

$$\mathbb{P} = \varepsilon_0 \chi_e \mathbb{E}$$

χ_e: elektrisk suceptibilitet

Då fås $\mathbb{D} = \varepsilon_0 \underbrace{(1 + \chi_e)}_{=\varepsilon_r\text{, relativa permabiliteten}} \mathbb{E} = \varepsilon_0 \varepsilon_r \mathbb{E}$

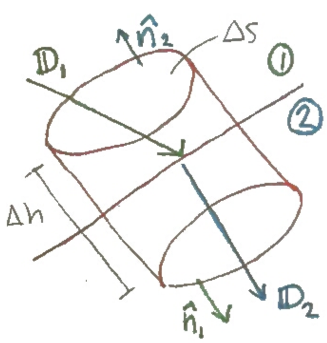
Randvillkor för elektrostatiska fält 3.9



$$0 = \int_{abcd} \mathbb{E} \cdot d\mathbb{l} = \mathbb{E}_1 \cdot \Delta w + \mathbb{E}_2 \cdot (-\Delta w) =$$

$$= E_{1t} \cdot \Delta W - E_{2t} \Delta W = 0 \quad , \quad E_{1t} = E_{2t}$$

Normalkomponenten



$$\oint_S \mathbb{D} \cdot d\mathbb{S} = (\mathbb{D}_1 \cdot \hat{n}_2 + \mathbb{D}_2 \cdot \hat{n}_1) \Delta S =$$

$$= \hat{n}_2 \cdot (\mathbb{D}_1 - \mathbb{D}_2) \Delta S = \rho_S \Delta S$$

så $(\mathbb{D}_1 - \mathbb{D}_2) \cdot \hat{n}_2 = \rho_S$

$$D_{1n} - D_{2n} = \rho_S$$

Storgruppsövning 30/10-13

Electric field intensity and Gauss's law

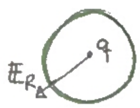
E: defined as the force per unit charge $\mathbb{E} = \lim_{q \to 0} \frac{F}{q} \ (\frac{V}{m})$

Two fundamental postulates in electrostatic (free space):

$\nabla \cdot \mathbb{E} = \frac{\rho}{\varepsilon_0} \Longrightarrow \oint_S \mathbb{E} \cdot d\mathbb{S} = \frac{Q}{\varepsilon_0}$ (Gauss's law)

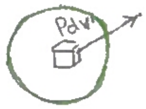
$\nabla \times \mathbb{E} = 0 \Longrightarrow \oint_S \mathbb{E} \cdot d\ell = 0$ (Kirchoff's voltage law)

Coulomb's law:

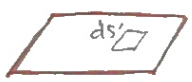


$$\mathbb{E} = a\hat{R} \frac{q}{4\pi\varepsilon_0 R^2}$$

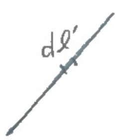
$$\mathbb{E} = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{N} \frac{q_k (\mathbb{R} - \mathbb{R}'_k)}{|\mathbb{R} - \mathbb{R}'_k|^3}$$ electric field due to a system of discrete charges.



$[\rho] = (\frac{c}{m^3})$ $\quad \mathbb{E} = \frac{1}{4\pi\varepsilon_0} \int_{V'} \rho \frac{\mathbb{R}}{|\mathbb{R}|^3} dV' = \frac{1}{4\pi\varepsilon_0} \int_{V'} \vec{aR} \frac{\rho}{|\mathbb{R}|^2} dv'$



$[\rho_s] = (\frac{c}{m^2})$ $\quad \mathbb{E} = \frac{1}{4\pi\varepsilon_0} \int_{S'} \vec{aR} \frac{\rho_s}{|\mathbb{R}|^2} ds'$



$[\rho_\ell] = (\frac{c}{m})$ $\quad \mathbb{E} = \frac{1}{4\pi\varepsilon_0} \int_{\ell'} \vec{aR} \frac{\rho_\ell}{|\mathbb{R}|^2} d\ell'$

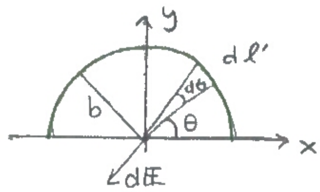
Gauss's law:

$\oint_S \mathbb{E} \cdot d\mathbb{S} = \frac{Q}{\varepsilon_0}$ outward flux of the electric field over any closed surface equals to the total charge in surface over ε_0.

It is useful to find $\mathbb{E}$ in symmetric problems.

Problem 3.8

line charge ρ_ℓ



$$\rho_\ell = dq/d\ell$$

$$dq = \rho_\ell \cdot d\ell' = \rho_\ell \cdot b \cdot d\theta$$

$$d\mathbf{E} = a\hat{R}\,\frac{dq}{4\pi\varepsilon_0 R^2} = -\hat{r}\,\frac{\rho_\ell \cdot b \cdot d\theta}{4\pi\varepsilon_0 b^2}\quad,\quad \hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta$$

$$d\mathbf{E} = (\hat{x}\cos\theta + \hat{y}\sin\theta)\,\frac{-\rho_\ell \cdot d\theta}{4\pi\varepsilon_0 b} = \hat{x}\,dE_x + \hat{y}\,dE_y$$

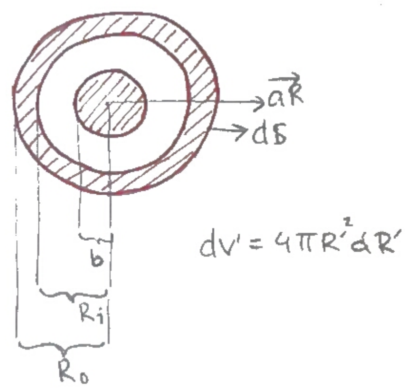
$$E_x = \int_{\theta=0}^{\pi} dE_x = 0$$

$$E_y = \int_{\theta=0}^{\pi} dE_y = \int_0^{\pi} \frac{-\rho_\ell}{4\pi\varepsilon_0 b}\sin\theta\,d\theta = \frac{-\rho_\ell}{2\pi\varepsilon_0 b}$$

$$\mathbf{E} = \hat{y}E_y = -\hat{y}\,\frac{\rho_\ell}{2\pi\varepsilon_0 b}$$

Problem 3.11

$$\rho = \rho_0\left[1 - \frac{R^2}{b^2}\right]\quad,\quad 0 \le R < b$$



$$\oint_s \mathbf{E}\cdot d\mathbf{s} = \frac{Q_{in}}{\varepsilon_0} \Rightarrow \oint E_R(R)\cdot dS = \frac{Q_{in}}{\varepsilon_0}$$

$$E_R(R)\oint_s dS = \frac{Q_{in}}{\varepsilon_0}$$

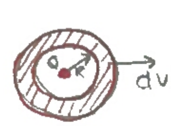
$$E_R(R)\,4\pi R^2 = \frac{Q_{in}}{\varepsilon_0}$$

$$E_R(R) = \frac{Q_{in}}{4\pi\varepsilon_0 R^2}$$

1) $0 \le R \le b$

$$Q_{in} = \int_0^R \rho\,dV' = \int_0^R \rho_0\left[1 - \frac{R'^2}{b^2}\right]4\pi R'^2\,dR' = 4\pi\rho_0\int_0^R\left[R'^2 - \frac{R'^4}{b^2}\right]dR' =$$

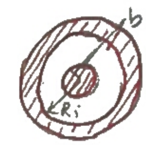
$$= 4\pi\rho_0\left[\frac{R^3}{3} - \frac{R^5}{5b^2}\right]$$



$$E_R(R) = \frac{Q_{in}}{4\pi\varepsilon_0 R^2} = \frac{4\pi\rho_0}{4\pi\varepsilon_0 R^2}\left[\frac{R^3}{3} - \frac{R^5}{5b^2}\right] = \frac{\rho_0}{\varepsilon_0}\left[\frac{R}{3} - \frac{R^3}{5b^2}\right]$$

2) $b \le R \le R_i$

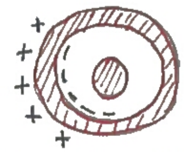
$$Q_{in} = Q_{in-1}\Big|_{R=b} = 4\pi \rho_0 \left[\frac{b^3}{3} - \frac{b^5}{5b^2}\right] = 4\pi \rho_0 b^3 \frac{2}{15}$$



$$E_{R2}(R) = \frac{4\pi \rho_0 b^3}{4\pi \varepsilon_0 R^2} \cdot \frac{2}{15} = \frac{\rho_0 b^3}{\varepsilon_0 R^2} \cdot \frac{2}{15}$$

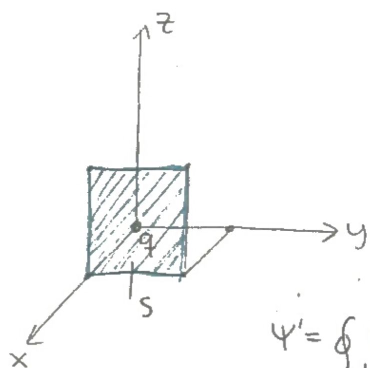
3) $R_i \le R \le R_o$

$$Q_{in} = 0 \Longrightarrow E_{R3}(R) = 0$$



4) $R \ge R_o \Longrightarrow Q_{in} = Q_{in2} \Longrightarrow E_{R4}(R) = E_{R2}(R) = \frac{2\rho_0 b^3}{\varepsilon_0 R^2 15}$

Problem 2.6



$$\Psi = \int_S \varepsilon_0 E \cdot dS$$

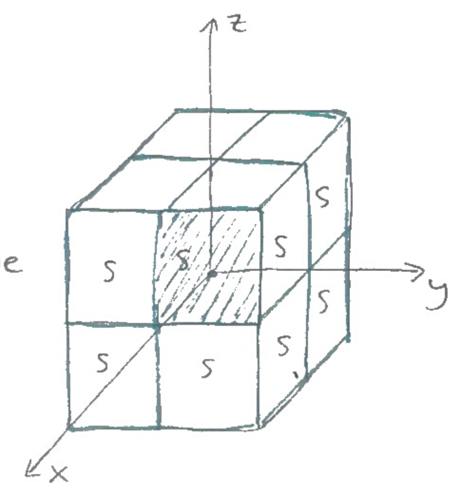
Make the problem symmetric.
Build a closed surface.

$$\Psi' = \oint_{S'_t} \varepsilon_0 E \cdot dS = q \quad (*)$$

$S_t = 6 \cdot 4S = 24S$, total area of cube

$$\Psi' = \oint_{S_t} \varepsilon_0 E \, dS = 24 \int_S \varepsilon_0 E \, dS = 24\Psi$$

$$(*) \Rightarrow 24\Psi = q \Longrightarrow \Psi = q/24$$



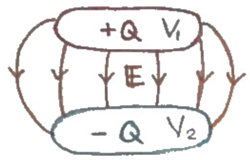
Föreläsning 1/11-13

Kapacitans 3.10 (ej 3.10.2, 3.10.3)

Definition kapacitans: $C = \frac{Q}{V}$, C är oberoende av Q och V, ty de är linjärt beroende.

Defintionen ok för ensam ledare. $V_\infty = 0$

Kondensator



Beräkna C

1- Placera $\pm Q$ på ledarna
2- Beräkna $\mathbb{E}$ från Q
3- Beräkna $V_{12} = V_1 - V_2 = \int_1^2 \mathbb{E} \cdot d\mathbb{l}$
4- $C = Q/V_{12}$

Metod 2

1- Ge ledarna potential V_1 och V_2
2- Finn $V(R)$
3- Beräkna $\mathbb{E} = -\nabla V$
4- Beräkna $Q_1 = \oint_{S_1} \varepsilon_0 \mathbb{E}\, d\mathbb{S}$
5- $C = Q_1/(V_1 - V_2)$
(se exempel 3.17 hemma)

Elektrostatisk energi 3.11

① $W_1 = Q_1 \cdot 0$
② $W_2 = Q_2 \cdot \frac{Q_1}{4\pi\varepsilon_0 R_{21}}$
③ $W_3 = Q_3 \cdot \left(\frac{Q_1}{4\pi\varepsilon_0 R_{31}} + \frac{Q_2}{4\pi\varepsilon_0 R_{32}}\right)$
⋮
ⓝ $W_n = Q_n \cdot \left(\frac{Q_1}{4\pi\varepsilon_0 R_{n1}} + \dots + \frac{Q_{n-1}}{4\pi\varepsilon_0 R_{n,n-1}}\right)$

$$\boxed{\begin{array}{l}\text{Har utgått från} \\ W_{mek} = q\,(V(P_2) - V(P_1)) \text{ och} \\ \text{har satt vår ref.pkt i } \infty \\ \Rightarrow W_{mek} = q \cdot V\end{array}}$$

Total energi: $W_e = \sum_{k=1}^{n} W_k$

Börja med Q_n
$W'_n = Q_n \cdot 0$
$W'_{n-1} = Q_{n-1}\left(\frac{Q_n}{4\pi\varepsilon_0 R_{n-1,n}}\right)$
⋮
$W'_1 = Q_1\left(\frac{Q_n}{4\pi\varepsilon_0 R_{1n}} + \frac{Q_{n-1}}{4\pi\varepsilon_0 R_{1,n-1}} + \dots + \right)$

Beräkna $2W_e = \sum_{k=1}^{n}(W_k + W_k') = \left\{\begin{array}{l}\text{har nu att alla termer i} \\ \text{summan är lika stora}\end{array}\right\} =$

$= Q_1V_1 + Q_2V_2 + Q_3V_3 + \dots + Q_nV_n \Longrightarrow W_e = \frac{1}{2}\sum_{k=1}^{n} Q_kV_k$

Generalisera:

$V_k \rightsquigarrow V(R)$

$Q_k \rightsquigarrow \rho(R)dV$ (se exempel 3.24 hemma)

$$\Rightarrow W_e = \frac{1}{2}\int_{V'} V(R)\rho(R)dV'$$

Alternativ form på energin:

$$W_e = \frac{1}{2}\int_{V'} V(R)\rho(R)dV' = \frac{1}{2}\int_{V'} V(\nabla\cdot\mathbb{D})dV' = \{\nabla\cdot(V\mathbb{D}) = V\cdot(\nabla\cdot\mathbb{D}) + \mathbb{D}\cdot\nabla V\} =$$

$$= \frac{1}{2}\int_{V'}(\nabla\cdot(V\mathbb{D}) - \mathbb{D}\cdot\nabla V)dV' = \underbrace{\frac{1}{2}\oint_{S'} V\mathbb{D}\cdot d\mathbb{S}}_{\substack{\to 0 \text{ då } R\to\infty \\ \text{ty } V\approx 1/R,\ D\approx 1/R^2 \\ dS\approx R^2}} + \frac{1}{2}\int_{V'}\mathbb{D}\cdot\mathbb{E}\,dV'$$

$$\Rightarrow \frac{1}{2}\int_{V'} V(R)\rho(R)dV' = \frac{1}{2}\int_{V'}\mathbb{D}\cdot\mathbb{E}\,dV'$$

<u>Energimetoder för kraftberäkning 3.11.2</u>

Coloumb's lag bra med fåtal laddningar

Istället kan vi relatera ändringar i elektrostatisk energi till kraft.

1. system av kroppar med fix laddning
2. system av ledande kroppar med fix potential.

se ex. 3.26 hemma

1. $dW = \mathbb{F}_Q\cdot d\mathbb{l}$ Mek. arbete utfört av systemet.

$$dW = -dW_e = \mathbb{F}_Q\cdot d\mathbb{l}$$

Förändring i elektrisk energi:

fix laddning.

från $W_e = \frac{1}{2}\sum_{k=1}^{n} Q_k V_k$ till $W_e = \frac{1}{2}\sum_{k=1}^{n} Q_k(V_k + dV_k)$

ekv. 2.8.8 $\quad dW_e = \nabla W_e \cdot d\mathbb{l}$

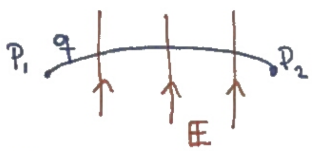
$$\Rightarrow -\nabla W_e\cdot d\mathbb{l} = \mathbb{F}_Q\cdot d\mathbb{l} \Rightarrow \mathbb{F}_Q = -\nabla W_e \quad \text{t.ex } (F_Q)_x = -\frac{\partial W_e}{\partial x}$$

Storgruppsövning 1/11-13

Electric potential

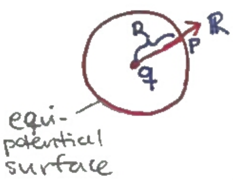
Identity: the curl of the gradient of any scalar field is zero.

$$\begin{cases} \nabla \times (\nabla V) = 0 \\ \nabla \times \mathbb{E} = 0 \end{cases} \Longrightarrow \mathbb{E} = -\nabla \underbrace{V}_{\text{scalar electric potential}}$$

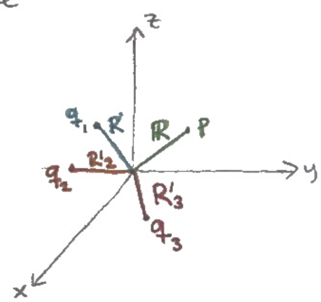


$$\frac{W}{q} = -\int_{P_1}^{P_2} \mathbb{E} \cdot d\mathbb{l} = \underbrace{V_2 - V_1}_{\substack{\text{potential difference} \\ \text{between } P_1 \text{ and } P_2}} \quad (V, \ J/C)$$

Potential at infinity is zero, $V_\infty = 0$.



$$V_R - \underbrace{V_\infty}_{=0} = V_R = -\int_\infty^R \underbrace{\mathbb{E}}_{\frac{q}{4\pi\varepsilon_0 R^2}} \cdot d\mathbb{l} = \frac{q}{4\pi\varepsilon_0 R}$$

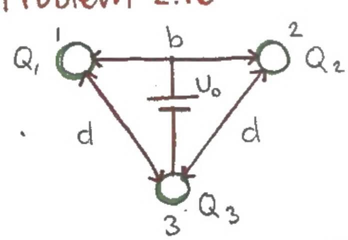


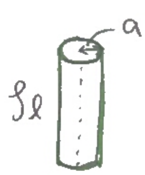
$$V_R = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{q_k}{|R - R'_k|}$$

$$V_R = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho}{R} \, dV', \quad \text{volume charge distribution}$$

$$V_R = \frac{1}{4\pi\varepsilon_0} \int_{S'} \frac{\rho_s}{R} \, dS' \quad \text{surface charge distribution}$$

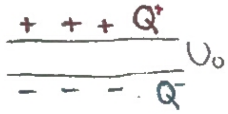
Problem 2.16



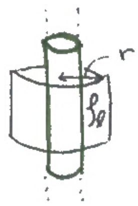


$$V_1 - V_3 = U_0$$

$$\begin{cases} Q_1 = Q_2 = Q \\ Q_3 = -2Q \end{cases}$$

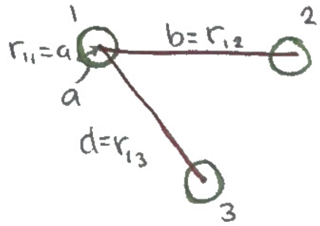


forts. →



Gauss law $\Longrightarrow$ $\mathbb{E} = \hat{a}_r E_r = \hat{a}_r \dfrac{\rho_\ell}{2\pi\varepsilon_0 r}$

$$V_r = -\int_\infty^r \mathbb{E}\cdot d\mathbb{l} = -\int_\infty^r \frac{\rho_\ell}{2\pi\varepsilon_0 r}\,dr = \frac{-\rho_\ell}{2\pi\varepsilon_0}\ln r$$

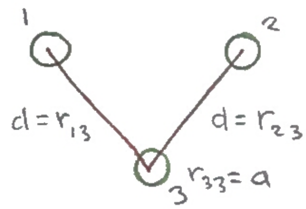


$$V_1 = V_{12} + V_{13} + V_{11} =$$

$$= \frac{-\rho_{\ell_2}}{2\pi\varepsilon_0}\ln r_{12} + \frac{-\rho_{\ell_3}}{2\pi\varepsilon_0}\ln r_{13} + \frac{-\rho_{\ell_1}}{2\pi\varepsilon_0}\ln r_{11}\,.$$

$$\Longrightarrow V_1 = \frac{-Q/\ell}{2\pi\varepsilon_0}\ln b + \frac{2Q/\ell}{2\pi\varepsilon_0}\ln d - \frac{Q/\ell}{2\pi\varepsilon_0}\ln a$$

$$V_1 = \frac{Q/\ell}{2\pi\varepsilon_0}\left[\ln d^2 - \ln(ab)\right] = \frac{Q/\ell}{2\pi\varepsilon_0}\ln\left(\frac{d^2}{ab}\right)$$



$$V_3 = V_{31} + V_{32} + V_{33} =$$

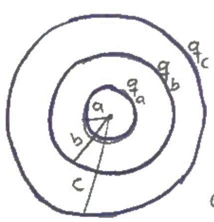
$$= \frac{-Q/\ell}{2\pi\varepsilon_0}\ln d - \frac{Q/\ell}{2\pi\varepsilon_0}\ln d + \frac{2Q/\ell}{2\pi\varepsilon_0}\ln a =$$

$$= \frac{Q/\ell}{2\pi\varepsilon_0}\left[\ln a^2 - \ln d^2\right] = \frac{Q/\ell}{2\pi\varepsilon_0}\ln\left(\frac{a^2}{d^2}\right)$$

$$V_1 - V_3 = U_0 \Longrightarrow \frac{Q/\ell}{2\pi\varepsilon_0}\left[\ln\left(\frac{d^2}{ab}\right) - \ln\left(\frac{a^2}{d^2}\right)\right] = \frac{Q/\ell}{2\pi\varepsilon_0}\ln\left(\frac{d^4}{a^3 b}\right) = U_0$$

$$\Longrightarrow Q = \frac{2\pi\varepsilon_0 V_0 \ell}{\ln\left(\frac{d^4}{a^3 b}\right)} \qquad \begin{cases} Q_1 = Q_2 = Q \\ Q_3 = -2Q \end{cases}$$

Problem 3.1



a) V_0 when $V_\infty = 0$?

b) b and c are connected by conductor, V_0?

a) $V_R - \underbrace{V_\infty}_{=0} = -\int_\infty^R \mathbb{E}\cdot d\mathbb{l} = V_0 = -\int_\infty^0 \mathbb{E}\cdot d\mathbb{l}$

Gauss's law: $\mathbb{E}_R(R) = \dfrac{Q_{in}}{4\pi\varepsilon_0 R^2}$

forts. →

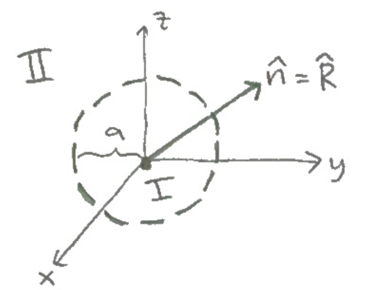
Electric flux density $\mathbb{D}$

$$\nabla\cdot\mathbb{E} = \frac{\rho}{\varepsilon_0} \quad \text{free space}$$

$$\nabla\cdot\mathbb{E} = \frac{\rho+\rho_P}{\varepsilon_0} \quad \text{Polarized dielectric} \overset{\rho_P=-\nabla\cdot\mathbb{P}}{\Longrightarrow} \nabla\cdot(\underbrace{\varepsilon_0 E+\mathbb{P}}_{\mathbb{D}}) = \rho \quad \text{free charge}$$

Problem 3.2

A dielectric sphere of radius a, has a constant polarization $\mathbb{P} = P\hat{R}$, V_0?



1. $V \Longrightarrow \mathbb{E} \overset{\text{Gauss}}{\Longrightarrow} Q_{in} \begin{cases}\rho_P\\ \rho_{PS}\end{cases}$
2. $V \Longrightarrow \begin{cases}\rho_P\\ \rho_{PS}\end{cases}$

methods...

$$\rho_P = -\nabla\cdot\mathbb{P} = -\frac{1}{R^2}\frac{\partial}{\partial R}(R^2P) = -\frac{2P}{R} \quad (R<a) \quad \text{volume charge density}$$

$$\rho_{PS} = \mathbb{P}\cdot\hat{n} = (P\hat{R})\cdot\hat{R} = P \quad \text{surface charge density}$$

1a) $\oint_S E\cdot ds = \frac{Q_{in}}{\varepsilon_0} \Longrightarrow E_R(R) = \frac{Q_{in}}{4\pi\varepsilon_0 R^2}$

$$\underset{0<R<a}{Q_{in}} = \int_{R'=0}^{R}\rho_P(R')dV' = \int_{R'=0}^{R} -\frac{2P}{R}4\pi R'^2 dR' = -8P\pi\int_{R'=0}^{R} R'dR' =$$

$$= -4\pi P\left[R'^2\right]_0^R = -4\pi PR^2$$

$R>a$ in II : $Q_{in}^{I}(R=a) + \oint_S \rho_{PS}\, dS' = -4\pi Pa^2 + P\overbrace{\oint_S dS'}^{4\pi a^2} = 0$

$$\Longrightarrow \begin{cases} E_{RI}(R) = \dfrac{-4\pi PR^2}{4\pi\varepsilon_0 R^2} = \dfrac{-P}{\varepsilon_0} & (R<a) \\[2ex] E_{RII}(R) = \dfrac{0}{4\pi\varepsilon_0 R^2} = 0 & (R>a) \end{cases}$$

1b) Use the Gauss's law

$$\oint_S \mathbb{D}\cdot d\mathbb{S} = Q \Longrightarrow D_R\oint_S ds = Q = 0 \Longrightarrow D_R = 0$$

$$0 = D = \varepsilon_0 E + P \Longrightarrow E = -\frac{P}{\varepsilon_0}, \quad E_R^{I}(R) = \frac{-P}{\varepsilon_0}, \quad E_R^{II}(R) = 0$$

$$V_0 = V(R=0) - \underbrace{V(R=\infty)}_{=0} = -\int_{\infty}^{0} E_R(R)\,dR =$$

$$= -\underbrace{\int_{\infty}^{a} E_R^{II}(R)\,dR}_{=0} - \int_{a}^{0} E_R^{I}(R)\,dR = \left[\frac{PR}{\varepsilon_0}\right]_{R=a}^{0} = \frac{-Pa}{\varepsilon_0}$$

2. $$V = \frac{1}{4\pi\varepsilon_0}\oint_s \frac{\rho_{ps}}{R}\,ds' + \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_p}{R}\,dv' =$$

$$= \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \frac{P}{4\pi\varepsilon_0 a}\, a^2 \sin\theta\, d\theta\, d\varphi + \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\int_{R=0}^{a} \frac{-2P}{4\pi\varepsilon_0 R^2} R^2 \sin\theta\, dR\, d\theta\, d\varphi =$$

$$= \frac{Pa}{4\pi\varepsilon_0}\, 2\pi \int_0^{\pi} \sin\theta\, d\theta - \frac{Pa}{2\pi\varepsilon_0}\cdot 2\pi \int_0^{\pi} \sin\theta\, d\theta =$$

$$= \frac{Pa}{2\varepsilon_0}\cdot 2 - \frac{Pa}{\varepsilon_0}\cdot 2 = \frac{-Pa}{\varepsilon_0}$$

Storgruppsövning 5/11-13

example 3.12

a positive point charge, Q, is at the center of a spherical dielectric shell (inner radius R_i and outer radius R_o)

E, V, D, P = ? as function of R.

$\begin{cases} \text{Spherical symmetry} \Rightarrow \text{Gauss} \Rightarrow E \\ V = -\int E \cdot dl, \quad D = \varepsilon E, \quad P = D - \varepsilon_0 E \end{cases}$

I) $R > R_o$, $\quad E_1 = \dfrac{Q}{4\pi\varepsilon_0 R^2}$, $\quad V_1 = \dfrac{Q}{4\pi\varepsilon_0 R}$, $\quad D_1 = \varepsilon_0 E = \dfrac{Q}{4\pi R^2}$, $\quad P_1 = 0$

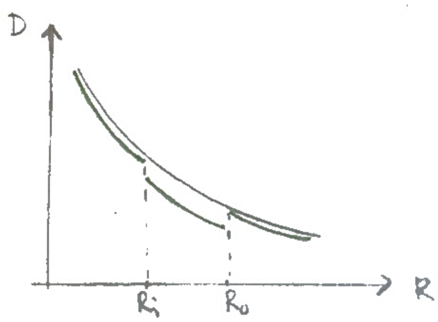
II) $R_i < R < R_o$, $\quad E_{R2} = \dfrac{Q}{4\pi\varepsilon_0\varepsilon_r R^2}$, $\quad D_2 = \varepsilon_0\varepsilon_r E = \dfrac{Q}{4\pi R^2}$, $\quad P_2 = \dfrac{Q}{4\pi R^2}\left(1 - \dfrac{1}{\varepsilon_r}\right)$

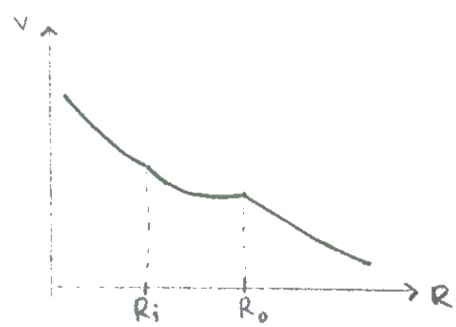
$$V_2 = -\int_\infty^R E \cdot dl = -\int_\infty^{R_o} E_1 \cdot dR - \int_{R_o}^R E_2 \, dR = V_1\Big|_{R=R_o} - \int_{R_o}^R \frac{Q}{4\pi\varepsilon R^2} dR =$$

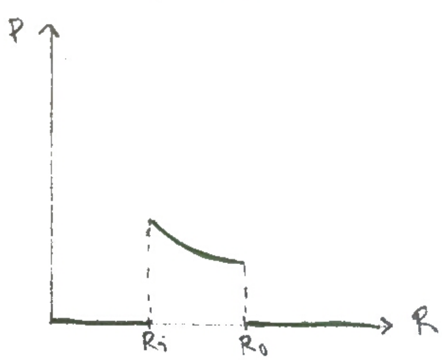
$$= \frac{Q}{4\pi\varepsilon_0}\left[\left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{R_o} + \frac{1}{\varepsilon_r R}\right]$$

III) $R < R_i$ $\quad E_3 = \dfrac{Q}{4\pi\varepsilon_0 R^2}$, $\quad D_3 = \varepsilon_0 E = \dfrac{Q}{4\pi R^2}$, $\quad P_3 = 0$

$$V_3 = V_2\Big|_{R=R_i} - \int_{R_i}^R E_3 \, dR = \frac{Q}{4\pi\varepsilon_0}\left[\left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{R_o} - \left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{R_i} + \frac{1}{R}\right]$$



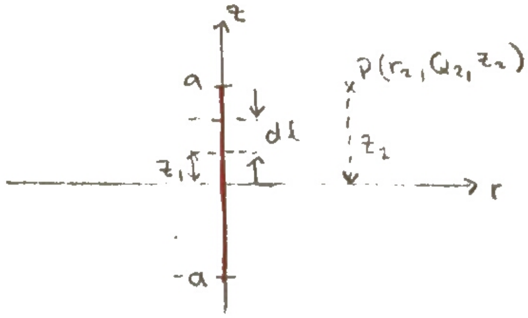




example 2.14

a homogenous line charge, ρ_ℓ, is located on z-axis, between $z=-a$ and $z=a$.

Find $E_r(r,z)$ and $E_z(r,z)$ of the electric field in point $P(r_2, \varphi_2, z_2)$



- not Gauss law!
- $dq = \rho_\ell \, d\ell = \rho_\ell \, dz_1$
- charge at $(\mathbb{R}_1 = z_1 \hat{z})$ gives -ve E-field at field point $\mathbb{R}_2$.

$\mathbb{R}_2 = r_2 \hat{r} + z_2 \hat{z}$ $\qquad$ $\mathbb{R}_{12} = \mathbb{R}_1 - \mathbb{R}_2 = r_2 \hat{r} + \hat{z}(z_2 - z_1)$

$dE = \dfrac{dq}{4\pi\varepsilon_0 R_{12}^2} \hat{R}_{12}$ $\qquad$ $R_{12} = \sqrt{r_2^2 + (z_2 - z_1)^2}$

Summing the contribution of dE from all dq:

$$\mathbb{E}(\mathbb{R}_2) = \frac{1}{4\pi\varepsilon_0} \int_{z=-a}^{a} \frac{\hat{r} r_2 + \hat{z}(z_2 - z_1)}{[r_2^2 + (z_2 - z_1)^2]^{3/2}} \underbrace{\rho_\ell \, dz_1}_{dq}$$

First we find the $\hat{r}$ component:

$$\hat{r} \cdot \mathbb{E}(\mathbb{R}_2) = \frac{\rho_\ell r_2}{4\pi\varepsilon_0} \int_{-a}^{a} \frac{1}{[r_2^2 + (z_2 - z_1)^2]^{3/2}} dz_1 = \left\{ \begin{array}{l} \text{subs. } \xi = z_2 - z_1 \\ z_1 = -a \Rightarrow \xi = z_2 + a \\ z_1 = a \Rightarrow \xi = z_2 - a \\ d\xi = -dz_1 \end{array} \right\} =$$

$$= -\frac{\rho_\ell r_2}{4\pi\varepsilon_0} \int_{z_2+a}^{z_2-a} \frac{1}{[r^2 + \xi^2]^{3/2}} d\xi = \left\{ \begin{array}{l} \int \frac{dx}{u^{3/2}} = \frac{x}{b\sqrt{u}}, \quad u = ax^2 + b \\ a = 1, \ b = r_2^2, \ x = \xi \end{array} \right\} =$$

$$= -\frac{\rho_\ell r_2}{4\pi\varepsilon_0} \left[\frac{\xi}{r_2^2 \sqrt{r_2^2 + \xi^2}} \right]_{z_2+a}^{z_2-a} = -\frac{\rho_\ell r_2}{4\pi\varepsilon_0 r_2^2} \left[\frac{z_2 - a}{\sqrt{r_2^2 + (z_2 - a)^2}} - \frac{z_2 + a}{\sqrt{r_2^2 + (z_2 + a)^2}} \right] =$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0 r_2} \left[\frac{z_2 + a}{\sqrt{r_2^2 + (z_2 + a)^2}} - \frac{z_2 - a}{\sqrt{r_2^2 + (z_2 - a)^2}} \right] \Rightarrow E_r \text{ component of P.}$$

Find the $\hat{z}$ component:

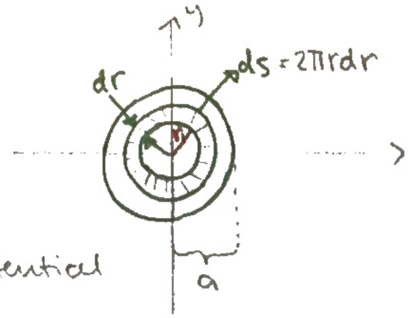
$$\hat{z} \cdot \mathbb{E}(\mathbb{R}_2) = \frac{-\rho_\ell}{4\pi\varepsilon_0} \int_{z_1=-a}^{a} \frac{z_2 - z_1}{[r_2^2 + (z_2 - z_1)^2]^{3/2}} dz_1 = \{ \text{subs } \xi = z_2 - z_1 \} =$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \int_{z_2+a}^{z_2-a} \frac{\xi}{[r_2^2 + \xi^2]^{3/2}} d\xi, \text{ p.s.s} \Rightarrow E_z = \frac{\rho_\ell}{4\pi\varepsilon_0} \left[\frac{1}{\sqrt{r_2^2 + (z_2 - a)^2}} - \frac{1}{\sqrt{r_2^2 + (z_2 + a)^2}} \right]$$

example 2.11

a thin circular metal disk of radius a, is located at very large distance from other bodies. A charge Q is distributed as a surface charge density on each side of the disk.

$\rho_s(r) = \dfrac{Q}{4\pi a\sqrt{a^2-r^2}}$



Find the potential of metal disk if $V(\infty)=0$.

- Potential is the same all over the disk.
- we compute the potential at center (V_0).

$$V = \int_S \frac{dq}{4\pi\varepsilon_0 R_{12}}$$

$$\begin{cases} \mathbf{R}_1 = \hat{r}\, r_1 \\ \mathbf{R}_2 = 0 \end{cases} \Rightarrow \quad \mathbf{R}_{12} = -\hat{r}\, r \ , \ R_{12} = r$$

$$dq = (\rho_s\, dS)\cdot 2 = \left(\frac{Q}{4\pi a\sqrt{a^2-r^2}}\cdot 2\pi r\, dr\right)\cdot 2$$, is charge on two sides.

$$V(R_2) = \int_S \frac{1}{4\pi\varepsilon_0 R_{12}}\, dq = \int_{r_1=0}^{a} \frac{1}{4\pi\varepsilon_0 \cancel{r_1}}\cdot \frac{Q\, \cancel{r_1}}{a\sqrt{a^2-r_1^2}}\, dr_1 = \frac{Q}{4\pi\varepsilon_0 a}\int_0^a \frac{1}{\sqrt{a^2-r_1^2}}\, dr_1 =$$

$$= \begin{Bmatrix} \displaystyle\int \frac{dx}{\sqrt{\tilde{b}-\tilde{a}x^2}} = \frac{1}{\sqrt{\tilde{a}}}\arcsin\left(x\sqrt{\frac{\tilde{a}}{\tilde{b}}}\right) \\ \tilde{b} = a^2 ,\ \tilde{a} = 1 ,\ x = r_1 \end{Bmatrix} = \frac{Q}{4\pi\varepsilon_0 a}\left[\arcsin\left(\frac{r_1}{a}\right)\right]_0^a =$$

$$= \frac{Q}{4\pi\varepsilon_0 a}\Big[\underbrace{\arcsin(1)}_{=\pi/2} - \underbrace{\arcsin(0)}_{=0}\Big] = \frac{Q}{4\cancel{\pi}\varepsilon_0 a}\cdot\frac{\cancel{\pi}}{2} = \frac{Q}{8\varepsilon_0 a}$$

Föreläsning 6/11-13

Poissons och Laplace ekv. 4.2

$$\left.\begin{array}{l} \nabla\cdot\mathbf{D} = \rho \Rightarrow \nabla(\varepsilon\mathbf{E}) = \rho \\ \nabla\times\mathbf{E} = 0 \Rightarrow \mathbf{E} = -\nabla V \end{array}\right\} \Rightarrow \nabla\cdot(\varepsilon\nabla V) = -\rho$$

Om ε är konstant i rummet: $\nabla^2 V = -\frac{\rho}{\varepsilon}$, Poissons ekv.

I områden utan laddning: $\nabla^2 V = 0$, Laplace ekv.

exempel 4.1 Plattkondensator
4.2 Sfärisk laddning

Entydighetssatsen 4.3

Med givna randvillkor är lösningen unik.

Bevis

Antag motsatsen, visa att motsatsen ej är sann.

Antag $\nabla^2 V_1 = \frac{\rho_1}{\varepsilon}$, $\nabla^2 V_2 = \frac{\rho_1}{\varepsilon}$

Bilda skillnaden $V_d = V_1 - V_2$

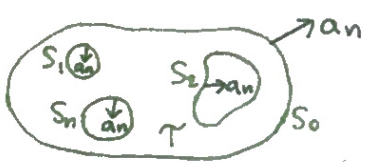
Då gäller $\nabla^2 V_d = 0$

Randvillkor: $V_d = 0$ på ledande ytor

$\frac{\partial V_d}{\partial n} = 0$ på isolerade ytor

Använd vektoridentitet: $\nabla(V_d \nabla V_d) = \underbrace{V_d \cdot \nabla^2 V_d}_{=0} + \nabla V_d \cdot \nabla V_d$

Integrera över volym:



$$\oint_S V_d \nabla V_d \cdot a_n \, dS = \int_\tau |\nabla V_d|^2 \, dV \quad \text{där } S = S_0 + S_1 + \ldots + S_n$$

$V_d = 0$ eller $\frac{\partial V_d}{\partial n} = 0 \Rightarrow \nabla V_d \cdot a_n = 0$, gäller för $S_1 \ldots S_n$

För S_0: $R \rightarrow \infty$, $V_d \approx 1/R$, $\nabla V_d \approx 1/R^2$, $S_0 \approx R^2$

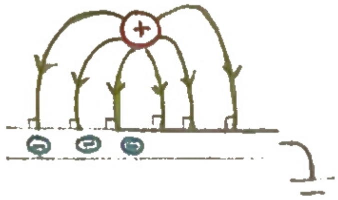
så $\oint_{S_0} V_d \nabla V_d \, dS \longrightarrow 0$ då $R \rightarrow \infty$

Så nu har vi $\int |\nabla V_d|^2 dV = 0$ men $|\nabla V_d| \geq 0$ överallt,

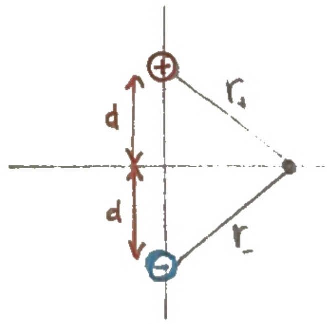
så måste $\nabla V_d = 0$, men $V_d = 0$

$\Rightarrow V_d \equiv 0$ (ty $\nabla V_d = 0 \Rightarrow V_d \equiv$ konstant)

Speglingsmetoden 4.4



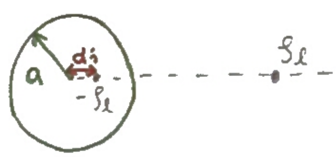
jordat ledande plan



$$V = \frac{Q}{4\pi\varepsilon}\frac{1}{r_+} + \frac{-Q}{4\pi\varepsilon}\frac{1}{r_-} = 0,$$

ty $r_+ = r_-$

Parallell linjeladdning utanför ledande cylinder

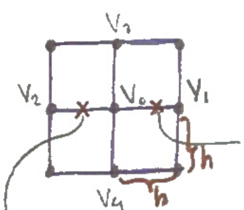


Vi har sett tidigare potential från två linjeladdningar: $V(r,\phi) = \frac{\rho_\ell}{2\pi\varepsilon_0}\ln\left(\frac{r^-}{r^+}\right)$

Måste gälla att $V_{cyl} = V(a,0) = V(a,\pi) = k$

$$\frac{\rho_\ell}{2\pi\varepsilon_0}\ln\left(\frac{a-d_i}{d-a}\right) = \frac{\rho_\ell}{2\pi\varepsilon_0}\ln\left(\frac{a+d_i}{d+a}\right) \Rightarrow \frac{a-d_i}{d-a} = \frac{a+d_i}{d+a} \Rightarrow d_i = \frac{a^2}{d}$$

Diskretisering av ∇^2 i rektangulära koordinater (ej i Cheng)



$$\left(\frac{\partial V}{\partial x}\right)_1 = \frac{V_1 - V_0}{h}$$

$$\left(\frac{dV}{dx}\right)_2 \approx \frac{V_0 - V_2}{h}$$

$$\nabla^2 V = 0$$

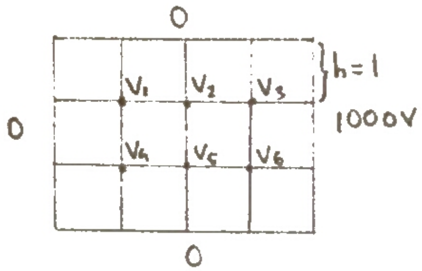
$$\left(\frac{\partial^2 V}{\partial x^2}\right)_0 = \frac{\left(\frac{\partial V}{\partial x}\right)_1 - \left(\frac{\partial V}{\partial x}\right)_2}{h} \approx \frac{V_1 + V_2 - 2V_0}{h^2}$$

$$\text{2-D: } (\nabla^2 V)_0 = \left(\frac{\partial^2 V}{\partial x^2}\right)_0 + \left(\frac{\partial^2 V}{\partial y^2}\right)_0 =$$

$$= \frac{V_1 + V_2 + V_3 + V_4 - 4V_0}{h^2} \quad (=0)$$

$$\Longrightarrow V_0 = \frac{1}{4}\left(V_1 + V_2 + V_3 + V_4\right)$$

exempel



Symmetri ger: $V_1 = V_4$
$V_2 = V_5$
$V_3 = V_6$

$$V_1 = \frac{1}{4}\left(V_2 + 0 + 0 + \overset{=V_1}{V_4}\right)$$

$$V_2 = \frac{1}{4}\left(V_3 + V_1 + \overset{=V_2}{V_5} + 0\right)$$

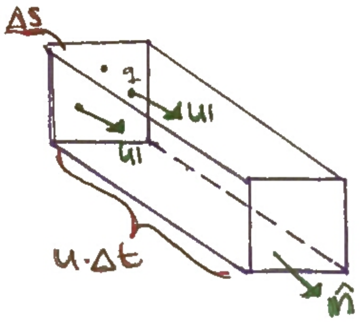
$$V_3 = \frac{1}{4}\left(V_2 + \overset{=V_3}{V_6} + 0 + 1000\right)$$

$$4\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1000 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1000 \end{bmatrix} \Rightarrow \begin{aligned} V_1 &= 47{,}6 \text{ V} \\ V_2 &= 142{,}9 \text{ V} \\ V_3 &= 381{,}0 \text{ V} \end{aligned}$$

Föreläsning 6/11-13

Elektrisk ström 5.1, 5.2



Låt volymen vara makroskopiskt liten.

Räkna laddningar som passerar gränsytan per tidsenhet.

N = laddningstäthet $[1/m^3]$

$$\Delta Q = Nq(\mathbf{u}\cdot\hat{n})\Delta t \; \Delta s$$

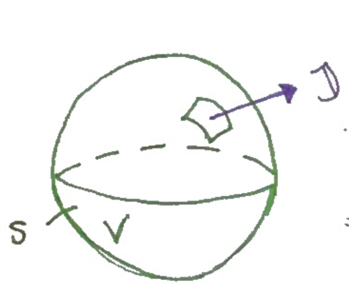
$$\Delta i = \frac{\Delta Q}{\Delta t} = Nq(\mathbf{u}\cdot\Delta\mathbf{s})$$

Fler laddningsbärare: $\Delta i = \sum_j (N_j q_j \mathbf{u}_j \cdot \Delta\mathbf{s})$

Definiera strömtäthet: $\mathbf{J} = \sum_j N_j q_j \mathbf{u}_j \quad [A/m^2]$

så $\Delta i = \mathbf{J}\cdot\Delta\mathbf{s}$, $di = \mathbf{J}\,d\mathbf{s}$

Kontinuitetsekvationen 5.4



Laddning kan ej förstöras

$$\Delta Q = -i\Delta t = -\Delta t \oint_S \mathbf{J}\,d\mathbf{s}$$

$$\Rightarrow -\frac{\partial Q}{\partial t} = \int \mathbf{J}\,d\mathbf{s}$$ Kontinuitetsekv. på integralform

Alternativ form: $\int_S \mathbf{J}\,d\mathbf{s} = \int_{V'} \nabla\cdot\mathbf{J}\,dV' = -\frac{\partial}{\partial t}\int_{V'} \rho\,dV'$

$$\int_{V_{godtycklig}} \underbrace{\left(\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t}\right)}\,dV' = 0$$

$$\Rightarrow \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$$ Kontinuitetsekv. på punktform

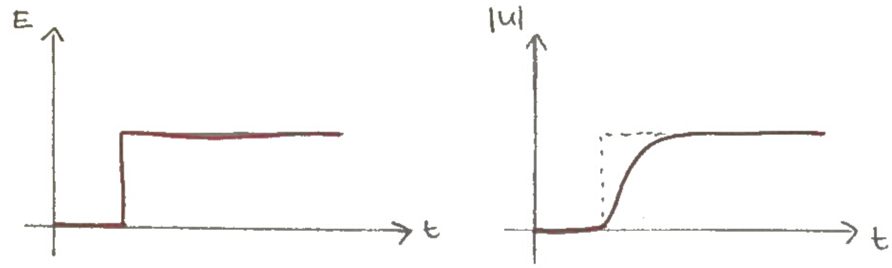
$\nabla\cdot\mathbf{J} = 0$ statik, likström

Ohms lag för metaller 5.2

Elektronernas rörelse beskrivs av

$$-e\mathbb{E} = m_e \frac{du}{dt} + m_e \nu u$$

Lösningen



lösning: $u(t) = \frac{-e\mathbb{E}}{m_e \nu}\left[1 - \exp(-\nu t)\right]$

Stationärtillstånd:

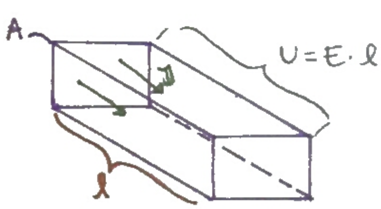
$$u = \frac{-e\mathbb{E}}{m_e \nu} = -\mu_e \cdot \mathbb{E}$$

μ_e: mobilitet

Sätt in i def. av ström: $J = -eNu = \frac{Ne^2}{m_e \nu}\mathbb{E} = \overset{\text{ohms lag}}{\sigma \mathbb{E}} = eN\mu_e \mathbb{E}$

Ohms lag: $J = \sigma \mathbb{E}$

Som i kretsen



$$I = J \cdot \Delta S = \sigma E \cdot \Delta S \cdot \frac{l}{l} = \frac{\sigma \Delta S}{l} \underbrace{E \cdot l}_{U}$$

$$I = \overset{1/R}{\frac{\sigma \Delta S}{l}} U \quad , \quad R = \frac{l}{\sigma \Delta S}$$

Relaxationstid 5.4

Vad händer med laddning på en metallplatta?
Börja med kont. ekv.:

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

Antag konstant σ:

$$\sigma \nabla \cdot \mathbb{E} = -\frac{\partial \rho}{\partial t} \Rightarrow \sigma \frac{\rho}{\varepsilon} + \frac{\partial \rho}{\partial t} = 0$$

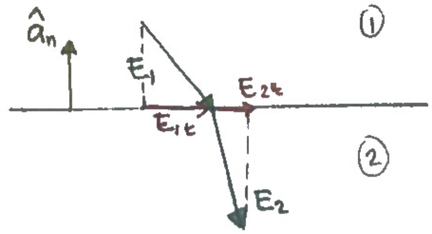
$$\rho = \rho_0 \exp\left(-\overset{1/\tau}{\frac{\sigma}{\varepsilon}} t\right) \quad , \quad \tau = \frac{\varepsilon}{\sigma} \text{ (relaxationstid)}$$

$\boxed{\text{kap 5.3 – EMK \& batterier} \quad \text{kap 5.6 – Randvillkor hos } J}$

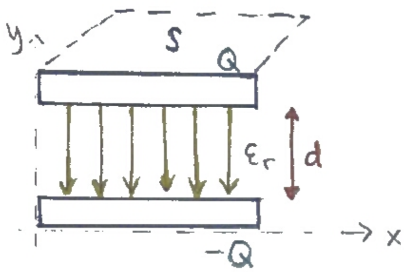
Storgruppsövning 8/11-13

- Boundary conditions for electrostatic field
- $E_{1t} = E_{2t}$
- $D_{1n} - D_{2n} = \rho_s$

  $\hat{a}_{n2} \cdot (D_1 - D_2) = \rho_s$

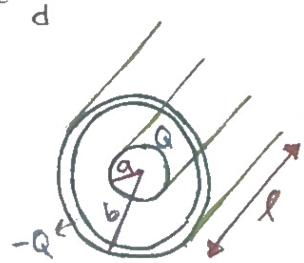


Capacitance:



$C = \frac{Q}{V_{12}}$ — charge on each conductor / — voltage difference between conductors.

$C = \varepsilon \frac{S}{d}$

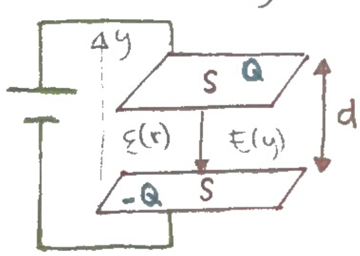


$C = \frac{Q}{V_{12}} = \frac{2\pi\varepsilon\ell}{\ln\left(\frac{b}{a}\right)}$

1. Choose a right coordinate system
2. Assume Q, -Q on conductors
3. Find E-field
4. $V_{12} = -\int E \, dl$
5. $C = \frac{Q}{V_{12}}$

P 3.30

The space between a parallel-plate capacitor of area S is filled with $\varepsilon_r(y)$.



$$\begin{cases} \varepsilon(y) = \varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{d} y \\ C = ? \end{cases}$$

- Cartesian coordinate

$$\nabla \cdot \underbrace{D}_{\varepsilon E} = \underbrace{\rho}_{zero} \Rightarrow \nabla(\varepsilon(y) E(y)) = \rho = 0$$

$$\varepsilon(y) E(y) = C_1 = \text{constant} \Rightarrow E(y) = \frac{C_1}{\varepsilon(y)}$$

$$V_0 = -\int_{y=0}^{y=d} E_y(y)\,dy = -\int_{y=0}^{y=d} \frac{C_1}{\varepsilon_1 + \frac{\varepsilon_2-\varepsilon_1}{d} y}\,dy =$$

$$= -C_1 \left[\frac{d}{\varepsilon_2-\varepsilon_1} \ln\left(\frac{\varepsilon_2-\varepsilon_1}{d} y + \varepsilon_1 \right) \right]_{y=0}^{y=d} = -C_1 \frac{d}{\varepsilon_2-\varepsilon_1} \ln \frac{\varepsilon_2}{\varepsilon_1}$$

$$\Rightarrow C_1 = -\frac{V_0}{d} \frac{(\varepsilon_2-\varepsilon_1)}{\ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right)}$$

$$E(y) = -\frac{V_0}{d} \frac{\varepsilon_2-\varepsilon_1}{\ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right)} \cdot \frac{1}{\varepsilon(y)}$$

$$D_y = \varepsilon E_y(y) = -\frac{V_0}{d} \frac{\varepsilon_2-\varepsilon_1}{\ln\left(\frac{\varepsilon_2}{\varepsilon}\right)}$$

normal boundary condition: $D_{1n} - \cancel{D_{2n}} = \rho_s \Rightarrow D_y = \rho_s$

$$\Rightarrow Q = \rho_s \cdot S = D_y \cdot S$$

$$C = \frac{Q}{V_0} = \frac{D_y \cdot S}{V_0} = \frac{1}{d} \frac{\varepsilon_2-\varepsilon_1}{\ln\left(\frac{\varepsilon_2}{\varepsilon_1}\right)} \cdot S$$

$$C = \frac{Q}{V_0} \underset{\Leftarrow}{\Rightarrow} \rho_s \underset{\Leftarrow}{\Rightarrow} D_y = D_{12} \underset{\Leftarrow}{\Rightarrow} E_y \underset{\Leftarrow}{\Rightarrow} V_0 = -\int E \cdot dl$$

Electrostatic energy and forces

$$W_e = \frac{1}{2} \sum_{k=1}^{n} Q_k V_k \longrightarrow$$ Potential energy of a group of N discrete charge

$$W_e = \frac{1}{2} \int_{V'} \rho V dV' \longrightarrow$$ continuous charge dist.

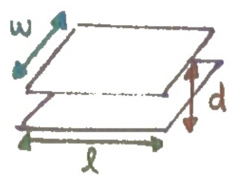
$$W_e = \frac{1}{2} \int D \cdot E \, dV$$ (in terms of E-field)

$$W_e = \frac{1}{2} C V^2 = \frac{Q^2}{2C} = \frac{1}{2} QV \quad (J)$$

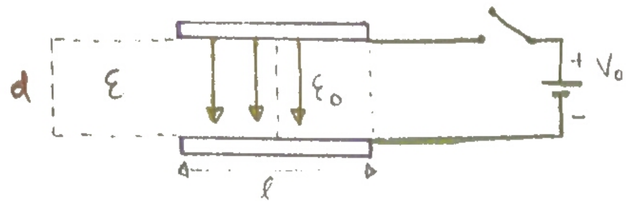
Electrostatic forces:

1. fixed charge system $F_Q = -\nabla W_e$
   - isolated
2. fixed potential system $F_V = +\nabla W_e$
   - connected to external sources

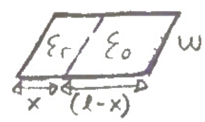
P 3.48



A parallell-plate capacitor, has a dielectric slab (ε) in the space between the plates. The dielectric slab is moved to a new position



a) when the switch closed.



$W_e = \frac{1}{2} Q_{in} V_0$, $\oint D \cdot dS = Q_{in}$, $D \to E$ *

$E_y = -\frac{V_0}{d} \Rightarrow D_y = \begin{cases} \text{air} & \varepsilon_0 E_y = (-\varepsilon_0 V_0)/d \\ \varepsilon E_y \text{ dielectric} & \varepsilon E_y = (-\varepsilon V_0)/d \end{cases}$

$\oint D \cdot ds = Q_{in} \Rightarrow Q_{in} = \frac{\varepsilon_0 \varepsilon_r V_0}{d} x w + \frac{\varepsilon_0 V_0}{d}(l-x) w =$

$= \frac{w \varepsilon_0 V_0}{d}\left(\varepsilon_r x + (l-x)\right)$ total charge

$W_e = \frac{1}{2} Q_{in} V_0 = \frac{1}{2} \frac{\varepsilon_0 V_0^2 w}{d}\left(l + (\varepsilon_r - 1)x\right)$

$F_V = +\nabla W_e = \frac{\partial W_e}{\partial x} = \frac{\varepsilon_0 V_0^2 w}{2d}(\varepsilon_r - 1)$, $C = \frac{Q_{in}}{V_0} = \frac{\varepsilon_0 w}{d}(\varepsilon_r x + (l-x))$

b) switch is open. (fixed charge system)

$W_e = \frac{Q^2}{2C} = \frac{Q^2}{2} \frac{d}{w} \frac{1}{\varepsilon x + \varepsilon_0 (l-x)}$

$F_Q = -\nabla W_e = \frac{Q^2}{2} \frac{d}{w} \frac{\varepsilon - \varepsilon_0}{[\varepsilon x + \varepsilon_0 (l-x)]^2}$

$Q = CV \to F_Q = F_V$

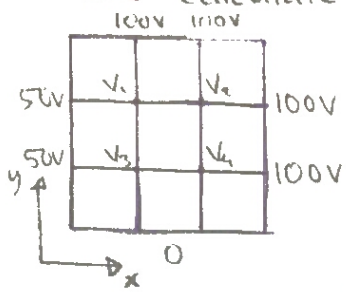
Poisson's and Laplace equation

$\nabla \cdot D = \rho$, $D = \varepsilon E$
$\nabla \times E = 0 \to E = -\nabla V$
$\Big\} \nabla^2 V = -\frac{\rho}{\varepsilon}$ Poissons eq.

Cartesian coord. system: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

when there is no free charge $\Rightarrow \nabla^2 V = 0$ Laplace eq.

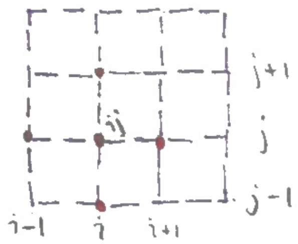
5.2 calculate the potential distribution numerically



$$\nabla^2 V = 0 \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \cancel{\frac{\partial^2 V}{\partial z^2}} = 0$$

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \cancel{\frac{d^2 V}{dz^2}} = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dV}{dx}\right) + \frac{d}{dy}\left(\frac{dV}{dy}\right) = 0$$



derivative can be approximated by differences between neighbouring points on a grid.

$$\frac{d}{dx} V = \lim_{h \to 0} \frac{(V_{i+1,j} - V_{i,j})}{h} \Rightarrow \frac{d}{dx}\left(\frac{dV}{dx}\right) = \lim_{h \to 0}\left(\frac{(V_{i+1,j} - V_{i,j}) - (V_{i,j} - V_{i-1,j})}{h^2}\right)$$

$$= \lim_{h \to 0}\left(\frac{V_{i+1,j} - 2_{i,j} + V_{i-1,j}}{h^2}\right)$$

$$\nabla^2 V = \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{h^2} + \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{h^2} = \frac{V_{i,j+1} + V_{i,j-1} + V_{i+1,j} + V_{i-1,j} - 4V_{i,j}}{h^2}$$

a) $V_{i,j+1} + V_{j,j-1} + V_{i+1,j} + V_{i-1,j} - 4V_{i,j} = 0$

$P_1: 100 + V_3 + V_2 + 50 - 4V_1 = 0$

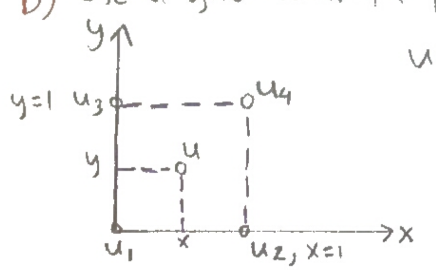
$P_2: 100 + V_4 + 100 + V_1 - 4V_2 = 0$

$P_3: V_1 + 0 + V_4 + 50 - 4V_3 = 0$

$P_4 = V_2 + 0 + 100 + V_3 - 4V_4 = 0$

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} -150 \\ -200 \\ -50 \\ -100 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 68,75 \\ 81,25 \\ 43,75 \\ 56,25 \end{bmatrix}$$

b) Use a grid with 49 points:

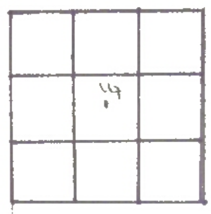


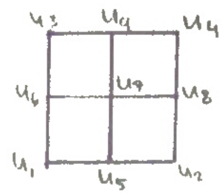
$$u(x,y) = u_1(1-x)(1-y) + u_2(1-y)x + u_3(1-x)y + u_4 xy$$

$0 < x < 1$, $0 < y < 1$

$x = 0,5$, $y = 0,5$ in the center

$$\Rightarrow u(0.5, 0.5) = \frac{u_1 + u_2 + u_3 + u_4}{4}$$





$u_7 = \frac{u_1 + u_2 + u_3 + u_4}{4}$ $\quad u_6 = \frac{u_3 + u_1}{2}$ $(x=0, y=0.5)$ $\quad u_8 = \frac{u_4 + u_2}{2}$ $(x=1, y=0.5)$

c) When we iterate more to find the exact potential
$\Rightarrow$ we use Gauss-Seidel method.

d) Analytic solution $V_0 = V_7 = 50 + \frac{200}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{\sinh(n\pi/2)}{\sinh(n\pi)} \sin\left(\frac{n\pi}{2}\right) = u_7$

≈ 62

Föreläsning 11/11-13

Joules lag 5.5

Arbete för att flytta en laddning q i fältet $\mathbb{E}$ sträckan Δl.

$$\Delta W = q\,\mathbb{E}\cdot\Delta l$$

Effekt: $\Delta P = \frac{\Delta W}{\Delta t} = q\cdot\mathbb{E}\cdot\frac{\Delta l}{\Delta t}$

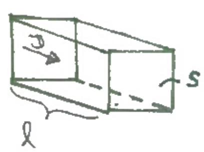
$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = q\,\mathbb{E}\cdot u$$

I en volym dV: $dP = \sum_i P_i = \mathbb{E}\cdot\left(\sum_i N_i q_i u_i\right)dV = \mathbb{E}\cdot\mathbb{J}\,dV = \frac{\mathbb{J}}{\sigma}\,\mathbb{J}\,dV = \frac{|\mathbb{J}|^2}{\sigma}dV$

olika laddningsbärare (i)

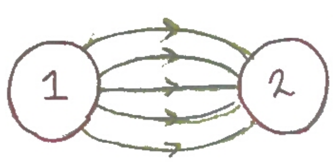
Effekttäthet: $\frac{dP}{dV} = \mathbb{E}\cdot\mathbb{J}$ $[W/m^3]$

För en volym V: $P = \int \mathbb{E}\cdot\mathbb{J}\,dV$, för en ledare med konstant tvärsnitt



$$P = \underbrace{\int \mathbb{E}\,dl}_{U}\;\underbrace{\int_S \mathbb{J}\,dS}_{I} = U\cdot I$$

Resistansberäkningar 5.7



Resistans: $R = \frac{\Delta V}{I}$

Konduktans: $G = \frac{1}{R}$

Jämför beräkning kapacitans: $C = \frac{Q}{\Delta V} = \frac{\oint \mathbb{D}\,dS}{\int_1^2 \mathbb{E}\cdot dl} = \varepsilon\,\frac{\oint \mathbb{E}\,dS}{\int_1^2 \mathbb{E}\,dl}$

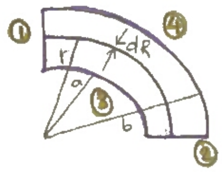
Resistansen: $\frac{1}{R} = G = \frac{I}{\Delta V} = \frac{\int \mathbb{J}\,dS}{\int_1^2 \mathbb{E}\cdot dl} = \sigma\,\frac{\int \mathbb{E}\cdot dS}{\int_1^2 \mathbb{E}\,dl}$

Resistans och kapacitans relaterade! $\frac{G}{C} = \frac{\sigma}{\varepsilon}$

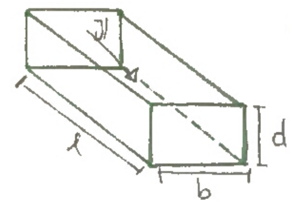
Seriekoppling: $R = \int dR$

Parallellkoppling: $G = \frac{1}{R} = \int dG_i$

exempel (jmf med ex 5.6)



tjocklek d



$R = \frac{1}{\sigma}\frac{\ell}{bd}$

Resistans ①–②: Summera parallella strömrör.

$$dR_{12} = \frac{1}{\sigma}\frac{\pi r}{2}\cdot\frac{1}{d\,dr}$$

$$dG_{12} = \frac{\sigma d\,dr}{\frac{\pi r}{2}} \quad , \quad G_{12} = \int_a^b \frac{\sigma d\,dr}{\pi r/2}$$

③–④: $dR_{34} = \dfrac{dr}{\sigma\frac{\pi r}{2}\cdot d} \Rightarrow R_{34} = \displaystyle\int_a^b \frac{dr}{\sigma\frac{\pi r}{2}\cdot d}$

Allmänt gäller: $R_{12}\cdot R_{34} = \left(\frac{1}{\sigma d}\right)^2 = \xi$ – ytresistivitet

Approximativ resistansberäkning (finns ej i boken)

Sats 1

En given ström som flyter i en isotrop ledare av godtycklig form fördelar sig så att totala värmeutvecklingen blir så liten som möjligt.

Sats 2

En given potentialskillnad...

Följdsats

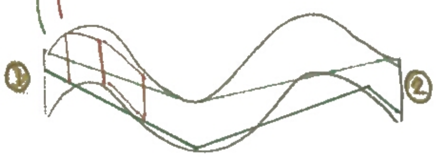
Varje ökning/minskning av resistiviteten någonstans i en ledare medför en ökning/minskning av totala resistansen.

Sats 3

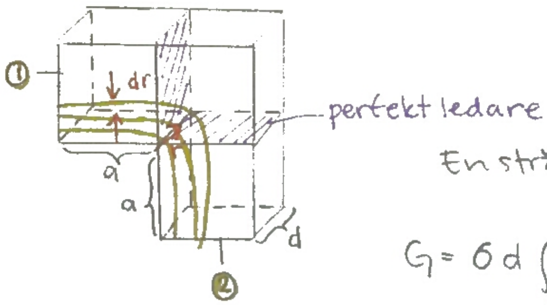
Varje approximativ strömfördelning ger för stort värde på den beräknade resistansen. Varje approximativ potentialfördelning ger för litet värde.

Räkneregel

Om vi lägger in tunna isolerande skikt som bildar strömrör får man för stor resistans.
Om man lägger in tunna oändligt gott ledande skikt bildar man ekvipotentialytor och man får för liten resistans.



exempel



Undre gräns

$$R_u = 2\frac{a}{\sigma a d} = \frac{2}{\sigma d}$$

Övre gräns

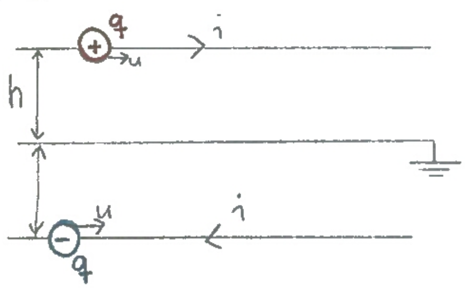
En strömbana. $dG = \sigma \frac{d\,dr}{2a+\frac{\pi}{2}r}$,

$$G = \sigma d\int_0^a \frac{dr}{2a+\frac{\pi}{2}r} = \frac{2\sigma d}{\pi}\ln\left(1+\frac{\pi}{4}\right)$$

$$R_ö = \frac{1}{G} = \frac{\pi}{2\sigma d}\,\frac{1}{\ln(1+\pi/4)}$$

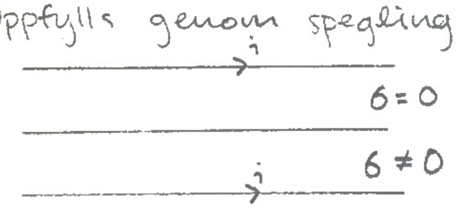
$$\Longrightarrow \quad \frac{2}{\sigma d} < R < \frac{2,71}{\sigma d}$$

Speglingsmetoden vid strömningsproblem



Spegling kan användas då normalkomponenten av strömtäthet är noll, $J\cdot\hat{n}=0$
(Gränsen mellan ledande/oledande material)

Uppfylls genom spegling med samma tecken och storlek.



Föreläsning 12/11-13

Det magnetiska fältet 6.1, 6.2

Kraft på stillastående laddning (elektrostatik): $\mathbb{F}_e = q\,\mathbb{E}(R)$

Kraft på laddning i rörelse med hastighet $\mathbb{u}$: $\mathbb{F} = \mathbb{F}_e + \mathbb{F}_m = q(\mathbb{E} + \mathbb{u} \times \mathbb{B})$

Postulat: $\nabla \cdot \mathbb{B} = 0$ (källfritt)

$\nabla \times \mathbb{B} = \mu_0 \cdot \mathbb{J}$ $\qquad [B] = Vs/m^2$, $\mu_0 = 4\pi \cdot 10^{-9}\, Vs/Am$

Vi vet att divergensen av en rotation $\equiv 0$

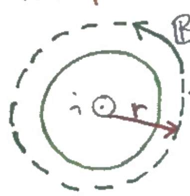
$\Rightarrow 0 \equiv \nabla \cdot (\nabla \times \mathbb{B}) = \mu_0 \nabla \cdot \mathbb{J} = 0$ (kont.ekv för likström)

Postulatet på integralform:

$$\oint_S \mathbb{B} \cdot d\mathbb{S} = 0$$

$$\oint \mathbb{B} \cdot d\ell = \mu_0 i \longleftarrow \text{Amperes lag}$$

exempel 6.1



(använd högerhandsregeln för att få ut riktning på $\mathbb{B}$-fältet)

Ampere slinga

omkrets för slingan

$\int \mathbb{B} \cdot d\ell = \mu_0 i \qquad B \cdot 2\pi r = \mu_0 i \qquad B = \dfrac{\mu_0 i}{2\pi r}$ (från lång rak tråd)

(ex 6.2 hemma)

Magnetisk vektorpotential 6.3

Vet att $\nabla \cdot (\nabla \times \mathbb{A}) \equiv 0$

så eftersom $\nabla \cdot \mathbb{B} = 0$

kan vi definiera $\mathbb{B} = \nabla \times \mathbb{A}$, $[A] = Vs/m$

För att definiera vektorn $\mathbb{A}$ behöver vi även

Studera nu $\nabla \times \mathbb{B} = \mu_0 \cdot \mathbb{J} \Rightarrow \nabla \times \nabla \times \mathbb{A} = \nabla(\nabla \cdot \mathbb{A}) - \nabla^2 \mathbb{A} = \mu_0 \mathbb{J}$

så vi väljer $\nabla \cdot \mathbb{A} = 0$

$-\nabla^2 \mathbb{A} = \mu_0 \mathbb{J}$ $\quad$ vektor Poisson (3st)

Lösning

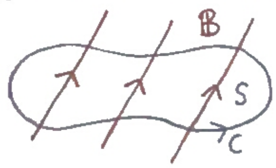
$\mathbb{A}(\infty)=0$

(Kom ihåg potentialuttrycket $V=\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho\, dV'}{R}$)

P.s.s $\mathbb{A}=\frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbb{J}}{R}dV'$

dvs $A_x=\frac{\mu_0}{4\pi}\int_{V'}\frac{J_x}{R}dV'$, $A_y=\dots$, $A_z=\dots$

Magnetiskt flöde 6.3



Definition

$\Phi=\int_S \mathbb{B}\cdot d\mathbb{S}=\int_S \nabla\times\mathbb{A}\,d\mathbb{S}=\oint_C \mathbb{A}\cdot d\mathbb{l}$ $(C=\partial S)$

↑ magnetiskt flöde

Biot-Savarts lag 6.4

$\mathbb{A}=\frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbb{J}(\mathbb{R}')}{R}dV'$

$\mathbb{B}=\nabla\times\mathbb{A}$ $\Longrightarrow \mathbb{B}=\frac{\mu_0}{4\pi}\int_{V'}\nabla\times\frac{\mathbb{J}(\mathbb{R}')}{R}dV' = (*)$

$\left\{\begin{array}{l}\text{Använd } \nabla\times f\,\mathbb{G}=f\,\nabla\times\mathbb{G}+\nabla f\times\mathbb{G} \\ \text{I vårt fall , } f=1/R \text{ , } \mathbb{G}=\mathbb{J}\end{array}\right\}$

$(*)=\frac{\mu_0}{4\pi}\int_{V'}\Big(\underbrace{\frac{1}{R}\nabla\times\mathbb{J}(\mathbb{R}')}+\nabla\left(\frac{1}{R}\right)\times\mathbb{J}(\mathbb{R}')\Big)dV'=\left\{\text{Använd } \nabla(1/R)=\frac{-\hat{\mathbb{R}}}{R^2}\right\}=$

($=0$ ty $\mathbb{J}=0$ i fältpkt $B(x',y',z')$ $\mathbb{J}(x,y,z)$)

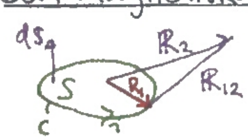
$=\frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbb{J}(\mathbb{R}')\times\hat{\mathbb{R}}}{R^2}dV'=\mathbb{B}$

Jämför elektrostatik: $\mathbb{E}=\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\mathbb{R}')\,\hat{\mathbb{R}}}{R^2}dV'$

P.s.s för ytström och linjeström
Fältbidrag från ett litet strömelement $d\mathbb{l}$
$dB=\frac{\mu_0 i}{4\pi}\cdot\frac{d\mathbb{l}'\times\hat{\mathbb{R}}}{R^2}$

(exempel 6.4, 6.5, 6.6 hemma)

Den magnetiska dipolen 6.5



$\mathbb{A}(\mathbb{R}_2)=\frac{\mu_0 i}{4\pi}\oint_C\frac{d\mathbb{l}_1}{R_{12}}=\left\{\oint_C\frac{d\mathbb{l}_1}{R_{12}}=\int_S d\mathbb{S}\times\nabla_1\left(\frac{1}{R_{12}}\right)\right\}$ Variant på Stokes.

$$\mathbb{A} = \frac{\mu_0 i}{4\pi} \int_S d\mathbb{S} \times \nabla_1 \left(\frac{1}{R_{12}}\right) = \frac{\mu_0 i}{4\pi} \int_S d\mathbb{S} \times \frac{\mathbb{R}_{12}}{R_{12}^3} = \dots =$$

$$\approx \frac{\mu_0 i}{4\pi} \int_S d\mathbb{S} \times \frac{\mathbb{R}_2}{R_2^3}$$

Definition

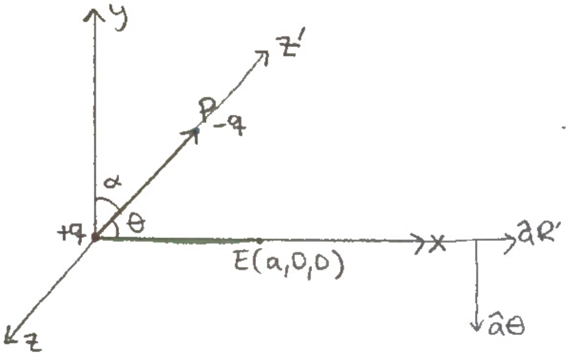
$$\mathbb{m} = i \int_S dS$$

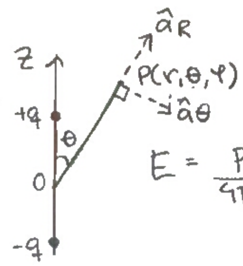
dipolmoment

Storgruppsövning 12/11-13

Electric dipole and dielectric material

2.12
A point dipole is located at the origin. The dipole moment P, lies in the x-y-plane. Fin E(a,0,0)





se 3.31 i boken

$$E = \frac{P}{4\pi\varepsilon_0 R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta)$$

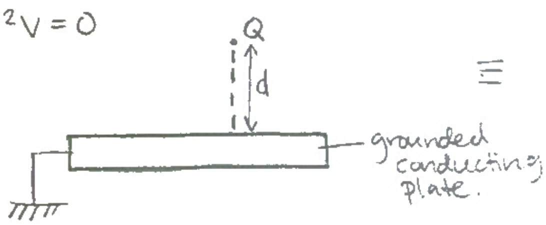
for this problem: $\hat{a}_R' = \hat{a}_x$, $\hat{a}_\theta' = -\hat{a}_y$, $\theta = \frac{\pi}{2} - \alpha$, $R = a$

$$\Rightarrow E = \frac{P}{4\pi\varepsilon_0 a^3}\left(\hat{a}_x 2\cos(\tfrac{\pi}{2}-\alpha) - \hat{a}_y \sin(\tfrac{\pi}{2}-\alpha)\right) =$$

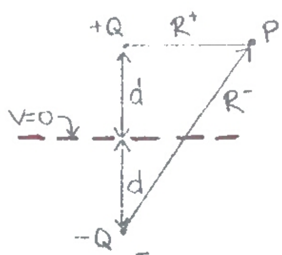
$$= \frac{P}{4\pi\varepsilon_0 a^3}(\hat{a}_x 2\sin\alpha - \hat{a}_y\cos\alpha)$$

Method of images

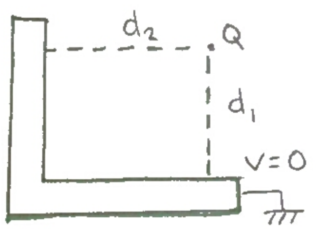
$\nabla^2 V = 0$



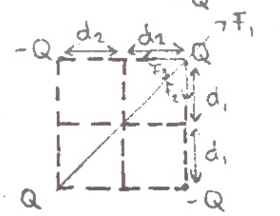
≡

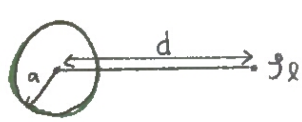


$$V_p = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R^+} - \frac{1}{R^-}\right)$$

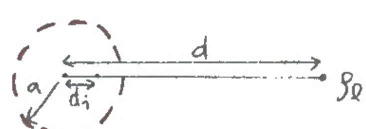


≡





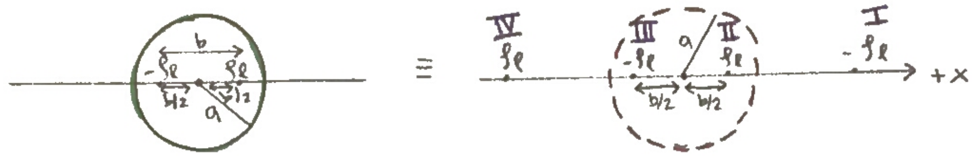
≡



$d_i = \frac{a^2}{d}$, $\rho_i = -\rho_\ell$

5.7 speglíng i cylinderyta:

inside a long metal tube (radius a), we have 2 thin metal wires passing through cylinder.



$\mathbb{E} = \hat{a}_r \dfrac{\rho_l}{2\pi\varepsilon_0 r}$ $\qquad d_i = \dfrac{2a^2}{b} \qquad d_i = \dfrac{a^2}{b/2} = \dfrac{2a^2}{b}$

$$\mathbb{E}^{II} = -\overset{I}{\frac{-\rho_l}{2\pi\varepsilon_0\left(\frac{2a^2}{b}-\frac{b}{2}\right)}} + \overset{III}{\frac{-\rho_l}{2\pi\varepsilon_0 b}} + \overset{IV}{\frac{\rho_l}{2\pi\varepsilon_0\left(\frac{2a^2}{b}+\frac{b}{2}\right)}} = 0 \qquad \frac{2a^2}{b} - \frac{b}{2}$$

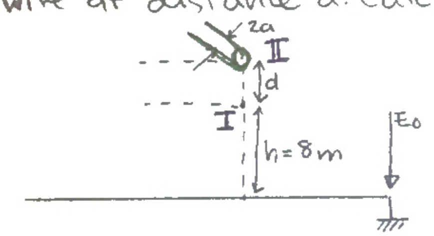
$$\mathbb{E}_x(x=b/2) = 0 \Rightarrow \frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{\frac{2a^2}{b}-\frac{b}{2}} + \frac{1}{\frac{2a^2}{b}+\frac{b}{2}} - \frac{1}{b}\right) = 0$$

$$\Rightarrow 16a^2b^2 - 16a^4 + b^4 = 0 \Rightarrow b = \pm\sqrt{4(\sqrt{5}-2)}\, a$$

5.5 Spegling i plan yta:

we have an isolated ~~uncharged~~ grounded thin wire at h = 8 m from ground* We add a protective wire of radius a in parallel with wire at distance d. Calculate the relative change in wire potential

* in E-field

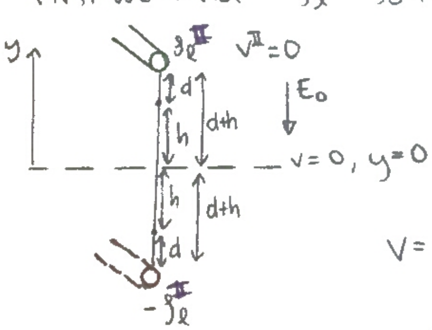


relative change in wire potential $= \dfrac{V^{I} - V_0}{V_0} = ?$ — V_0: potential without protective wire.

$$\begin{pmatrix} V^{II} = 0, & \rho_l^{II} \neq 0 \\ V^{I} \neq 0, & \rho_l^{I} = 0 \end{pmatrix}$$

a) $d = 1\,\text{m}$, $\dfrac{V^{I} - V_0}{V_0} = ?$

first we find ρ_l^{II} so that $V^{II} = 0$



$V(y=h+d) = V^{II} = 0$

$V(y=h+d) = E_0(h+d) + \ldots$

The electric potential at distance r from a line charge ρ_l is given as:

$$V = -\int_{r_0}^{r} E_r\,dr = -\frac{\rho_l}{2\pi\varepsilon_0}\int_{r_0}^{r}\frac{1}{r}\,dr = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{r_0}{r}\right) \qquad V_{r_0} = 0 \text{ reference point.}$$

$$V = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{r_0}{r^+}\right) - \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{r_0}{r^-}\right) = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{r^-}{r^+}\right)$$

$$V(y=h+d) = E_0(h+d) + \frac{\rho_\ell^{II}}{2\pi\varepsilon_0}\ln\left(\frac{2(h+d)}{a}\right) = 0$$

$$\Longrightarrow \rho_\ell^{II} = \frac{-2\pi\varepsilon_0 E_0(h+d)}{\ln\left(\frac{2(h+d)}{a}\right)}$$

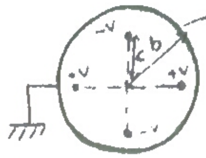
$$V^I = V(y=h) = E_0 h + \frac{\rho_\ell^{II}}{2\pi\varepsilon_0}\ln\left(\frac{2h+d}{d}\right)$$, substitute ρ_ℓ^{II}

$$V^I = \underbrace{E_0 h}_{V_0} - \frac{E_0(h+d)}{\ln\left(\frac{2(h+d)}{a}\right)}\ln\left(\frac{2h+d}{d}\right)$$

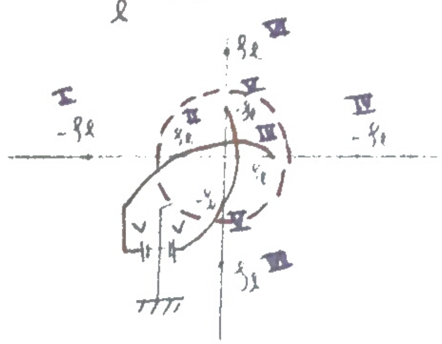
$$\Longrightarrow \frac{V^I - V_0}{V_0} = -\left(\frac{h+d}{h}\right)\cdot\frac{\ln\left(\frac{2h+d}{d}\right)}{\ln\left(\frac{2(h+d)}{a}\right)}$$

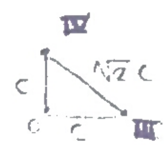
6.12

Spegling: cylinderyta, radius of wires $= a$.



Find the $\frac{C}{\ell}$ for two-conductor-cable.





$$V = \frac{\rho_\ell}{2\pi\varepsilon_0}\ln\left(\frac{r^-}{r^+}\right)$$

$$\frac{C}{\ell} = \frac{Q/V_{12}}{\ell} = \frac{Q/\ell}{V_{12}} = \frac{\rho_\ell}{2V^{III}} = \frac{\rho_\ell}{V^{III}-(-V^{III})}$$

$$V^{III} = V^{II} = \frac{\rho_\ell}{2\pi\varepsilon_0}\left[\ln\left(\frac{b^2/c+c}{2c}\right) + \ln\left(\frac{b^2/c-c}{a}\right) + \ln\left(\frac{\sqrt{2}\,c}{\sqrt{\left(\frac{b^2}{c}\right)^2+c^2}}\right)^2\right] =$$

$$= \frac{\rho_\ell}{2\pi\varepsilon_0}\ln\left[\frac{c(b^4-c^4)}{a(b^4+c^4)}\right]$$

$$\frac{C}{\ell} = \frac{\rho_\ell}{2V^{III}} = 2\pi\varepsilon_0 \Big/ \ln\left[\frac{c(b^4-c^4)}{a(b^4+c^4)}\right]$$

Föreläsning 13/11-13

$$\mathbb{m} = i\int_S d\mathbb{S}$$

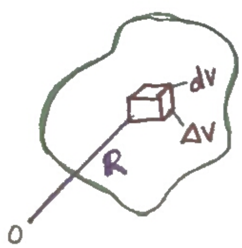
$$\mathbb{A}(R_2) = \frac{\mu_0}{4\pi}\,\mathbb{m} \times \frac{\mathbb{R}_2}{R_2^3}$$

I sfäriska koordinater med $d\mathbb{S}$ i z-led:

$$\mathbb{A}(R,\theta,\phi) = \hat{\phi}\,\frac{\mu_0}{4\pi}\,m\,\frac{\sin\theta}{R^2}$$

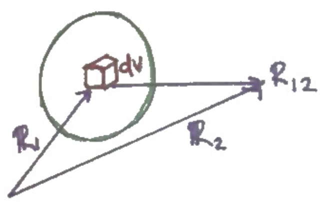
$$\mathbb{B} = \nabla \times \mathbb{A} = \frac{\mu_0 m}{4\pi R^3}\left(\hat{R}\,2\cos\theta + \hat{\theta}\sin\theta\right)$$

Magnetiseringsfältet $\mathbb{M}$ 6.6 (ej 6.6.1)



$$\mathbb{M} = \lim_{\Delta V \to 0} \frac{\sum_{k=1}^{n\Delta V} m_k}{\Delta V} \qquad [A/m]$$

eller $\dfrac{d\mathbb{m}}{dV} = \mathbb{M}$



$$d\mathbb{A}_m(\mathbb{R}_2) = \frac{\mu_0}{4\pi}\,\frac{\overbrace{\mathbb{M}\,dV}^{d\mathbb{m}} \times \mathbb{R}_{12}}{R_{12}^3}$$

$$\mathbb{A}_m(\mathbb{R}_2) = \int_{V'} \frac{\mu_0}{4\pi}\,\frac{\mathbb{M}\times\mathbb{R}_{12}}{R_{12}^3}\,dV' = \ldots =$$

$$= \frac{\mu_0}{4\pi}\int_{V'} \frac{\nabla_1 \times \mathbb{M}(\mathbb{R}_1)}{R_{12}}\,dV' + \frac{\mu_0}{4\pi}\oint_S \frac{\mathbb{M}(\mathbb{R}_1)}{R_{12}} \times d\mathbb{S}$$

Identifiera $\mathbb{J}_m(\mathbb{R}_1) = \nabla \times \mathbb{M}(\mathbb{R}_1) \longleftarrow$ Magnetiseringsströmtäthet

$\mathbb{J}_{ms}(\mathbb{R}_1) = \mathbb{M}\times\hat{n} \longleftarrow$ yt — " —

Tvärsnitt hos magnetiskt material:

H-fältet 6.7

Postulatet säger: $\frac{1}{\mu_0}\nabla\times\mathbf{B} = \mathbf{J}_{fria} + \mathbf{J}_m = (\mathbf{J} + \nabla\times\mathbf{M})$

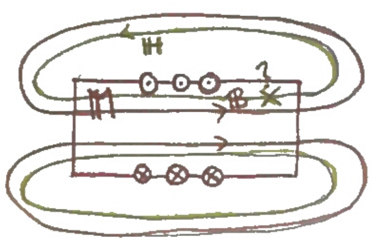
$\nabla\times\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}_{fri}$ Definiera: $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$, $\underbrace{\nabla\times\mathbf{H} = \mathbf{J}_{fri}}_{\text{postulatet med } \mathbf{H}\text{-fältet.}}$

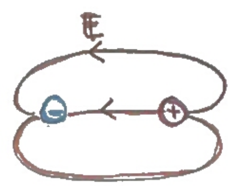
$\Longleftrightarrow \oint_C \mathbf{H}\cdot d\mathbf{l} = i_{fri}$

Låt $\mathbf{M} = \chi_m \mathbf{H}$

$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0\underbrace{(1+\chi_m)}_{\mu_r}\mathbf{H} = \mu_0\mu_r\mathbf{H}$

$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$





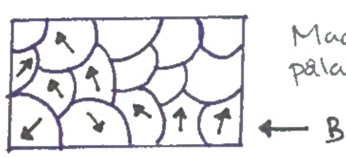
Randvillkor B & H 6.10

$\nabla\cdot\mathbf{B} = 0 \Longrightarrow \mathbf{B}_{1n} = \mathbf{B}_{2n}$

$\nabla\times\mathbf{H} = \mathbf{J} \Longrightarrow (\mathbf{H}_1 - \mathbf{H}_2)_{tang.} = \mathbf{j}_s \times \hat{\mathbf{n}}_2$ om $\mathbf{j}_s = 0 \Longrightarrow H_{1t} = H_{2t}$

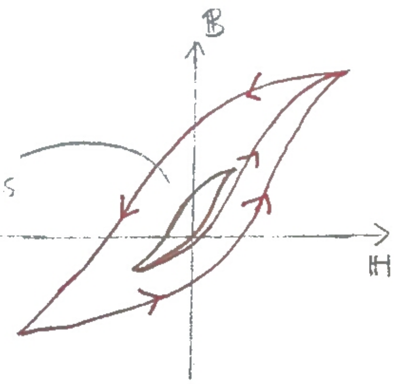
Hysteres värme 6.9

En feromagnet $\mu_r >> 1$



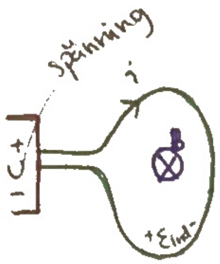
Magnetiska domäner utan pålagt fält (t.ex i gjutjärn)

Arean motsvarar värmeeffekt som utvecklas då vi omorienterar domänerna.



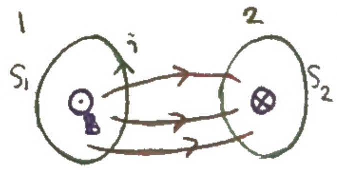
Induktans 6.11

Faradays induktionslag: $\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t}$ (postulat)



$$\varepsilon_{ind} = -\frac{d\Phi}{dt} = -\frac{d(\underbrace{L}_{\text{självinduktans}}\cdot i)}{dt} = -L\frac{di}{dt}$$

Ömsesidig induktans



(Har två slingor där ena inducerar i den andra, kallas ömsesidig induktans.)

Flöde från slinga 1 i 2.

$$\Phi_{12} = \int \mathbb{B}_1 \, d\mathbb{S}_2 = \underbrace{L_{12}}_{\text{ömsesidig induktans}} i_1$$

Flera varv i slingan

Länkat flöde: $\Lambda_{12} = \underbrace{N_2}_{\text{antal varv i slinga 2}} \int \mathbb{B}_1 d\mathbb{S}_2 = L_{12}\, i \quad \propto N_1 N_2\, i_1$

Självinduktans: $\Lambda_{11} = L_{11}\, i_1 \propto N_1^2\, i_1$

Storgruppsövning 13/11-13

Steady state currents

Conduction current: caused by motion of conduction electrons & holes
Convection current: caused by motion of electrons and ions.

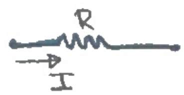
Current density: J (A/m^2), $J = Nqu$ ← velocity of charged carriers

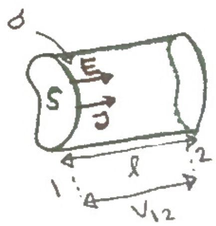
N – number of charge carrier per unit charge

$J = \rho u$ – convection current density, $J = \sigma E$ – σ: conductivity

$$I = \int_S J \cdot dS \quad (A)$$

Ohm's law: $V_{12} = RI$





$$\left.\begin{array}{l} V_{12} = E\ell \Rightarrow E = V_{12}/\ell \\ I = \int_S J\,dS = JS \Rightarrow \underbrace{J}_{\sigma E} = I/S \end{array}\right\} J = \sigma E = \sigma \frac{V_{12}}{\ell} = \frac{I}{S}$$

$$\Rightarrow \frac{V_{12}}{I} = \left(\frac{\ell}{\sigma S}\right) = R$$ – resistance for this conductor with cross section S.

$$\begin{cases} R_{seri.} = R_1 + R_2 + \dots \\ \frac{1}{R_{11}} = \frac{1}{R_1} + \frac{1}{R_2} + \dots = G_{11} = G_1 + G_2 + \dots \end{cases}$$

Equation of continuity: $\nabla \cdot J = \frac{-\partial \rho}{\partial t}$ (A/m^3)

Steady state current, DC current $\Rightarrow \frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla \cdot J = 0$

$\Rightarrow \oint_S J\,dS = 0 \Rightarrow \sum_j I_j = 0$ Kirchoff's current law

6.1
Stationär strömning

$\begin{cases} \sigma(R) \\ \varepsilon(R) \end{cases}$ varying in a medium, if a dc current pass $\Rightarrow$ we have a charge distribution (ρ).

Find a relation between σ & $\varepsilon \Rightarrow \rho = 0$.

forts. →

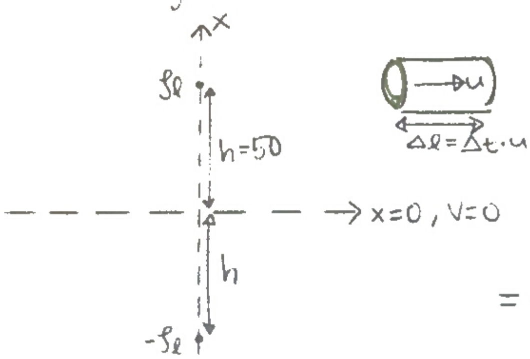
$$\begin{cases} J = \sigma E \\ D = \varepsilon E \end{cases} \qquad \begin{cases} \nabla \cdot D = \rho \\ \nabla \cdot J = -\dfrac{\partial \rho}{\partial t} \end{cases} \xrightarrow[\text{current}]{\text{dc}} \begin{cases} \nabla \cdot D = \rho \\ \nabla \cdot J = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \nabla \cdot (\varepsilon E) = \rho \\ \nabla \cdot (\sigma E) = 0 \end{cases} \xrightarrow[\varepsilon = \sigma \alpha \text{ (const.)}]{\text{assume}} \nabla (\alpha \sigma E) = \alpha \underbrace{\nabla \cdot (\sigma E)}_{=0} = \alpha \cdot 0 = \rho \Rightarrow \rho = 0$$

if $\varepsilon = \alpha \sigma$.

6.2

Dust charged particles are emitting from a chimney $h=50m$ from ground. Wind velocity: $5m/s$, they make a horizontal cylindrical charged cloud. (ρ_l)
Current: $100\ \mu A$, ground plane is a perfect conductor.
Find E on ground!



E of ρ_l $\rightarrow$ $E = \dfrac{\rho_l}{2\pi\varepsilon_0 r}\hat{r}$

In case of convection current:

$$J = \rho u, \quad i = \frac{\Delta Q}{\Delta t} = \frac{\rho_l \Delta l}{\Delta t} =$$

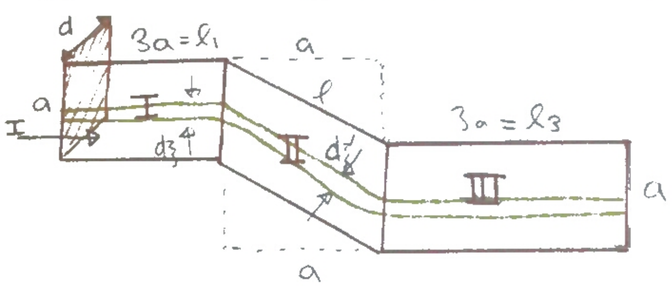
$$= \frac{\rho_l u \cdot \cancel{\Delta t}}{\cancel{\Delta t}} = \rho_l u \Rightarrow \rho_l = \frac{i}{u}$$

$$E_x(x=0) = \frac{\rho_l}{2\pi\varepsilon_0 h}(-\hat{x}) + \frac{-\rho_l}{2\pi\varepsilon_0 h}(\hat{x}) = \frac{-\rho_l}{\pi\varepsilon_0 h} = \frac{-i}{\pi\varepsilon_0 h u}\hat{x}$$

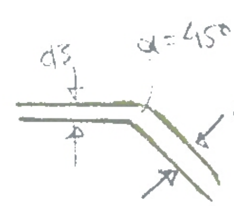
$$E_x(x=0) = \frac{-100 \cdot 10^{-6}}{\pi \cdot 8{,}85 \cdot 10^{-12} \cdot 50 \cdot 5} = 14 \left(\frac{kV}{m}\right)$$

6.11

Resistansberäkning direkt
use two approximation methods to find a lower and upper limit for the resistance between 2 electrons.



R_{min}?, R_{max}?



Upper bound: use non-physical current tubes $\rightarrow R_{max}$

$$d\xi' = d\xi \sin\alpha = d\xi \frac{1}{\sqrt{2}}$$

$$R = R^{I} + R^{II} + R^{III}$$

$$R^{I} = R^{III} = \frac{3a}{\sigma \cdot ad} = \frac{3}{\sigma d} = 3s \quad \left(s = \frac{1}{\sigma d}\right)$$

$$l = \sqrt{2}a \;, \quad R^{II} = \frac{l}{\sigma S}$$

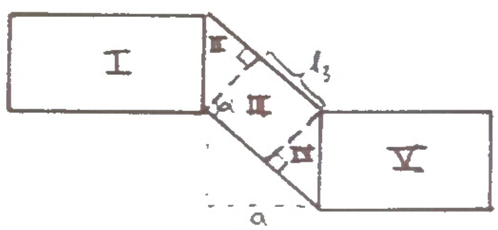
$$S = d\,d\xi' \;, \quad S = \frac{d}{\sqrt{2}}\,d\xi$$

$$dG^{II} = \frac{\sigma S}{l} = \frac{\sigma}{\sqrt{2}a}\left(\frac{d}{\sqrt{2}}\,d\xi\right) \Longrightarrow G^{II} = \int_{\xi=0}^{a} dG^{II} = \int_0^a \frac{\sigma d}{a \cdot 2}\,d\xi = \frac{\sigma a d}{2a} = \frac{\sigma d}{2}$$

$$R^{II} = \frac{2}{\sigma d} = 2s$$

$$R_{max} = 2R^{I} + R^{II} = 2 \cdot 3s + 2s = 8s = \frac{8}{\sigma d}$$

lower limit: use constant potential surface on dashed line



assume: $R^{II} = R^{IV} = 0 \quad (\sigma = \infty)$

$$R_{min} = R^{I} + R^{III} + R^{V}$$

$$R^{I} = R^{V} = \frac{l}{\sigma S} = 3s$$

$$l_3 = a\sin\alpha = a\sin 45^\circ = a/\sqrt{2}$$

$$h = a/\sqrt{2}$$

$$\Longrightarrow R^{III} = \frac{l_3}{\sigma S} = \frac{\cancel{a/\sqrt{2}}}{\sigma d \cancel{a/\sqrt{2}}} = \frac{1}{\sigma d} = s$$

$$R_{tot} = R_{min} = 2 \cdot 3s + s = 7s$$

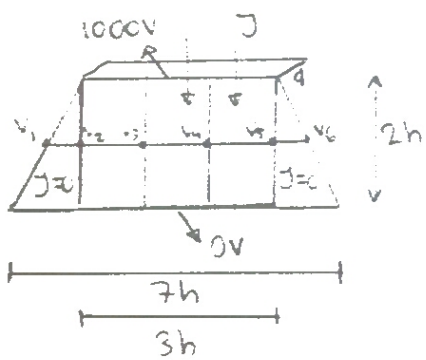
$$\Longrightarrow \quad 7s < R < 8s$$

6.20

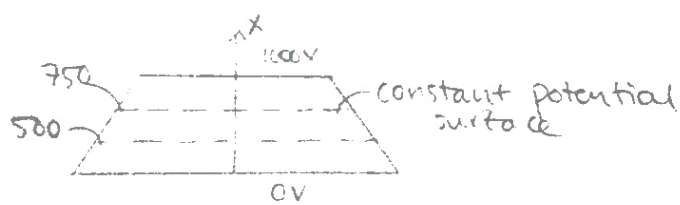
Numerisk beräkning

On a thin sheet two electrodes are fastened. $(\sigma = 5\ S/m, d = 0.1\,mm)$

Find the upper and lower resistance R!



$$R^{upper} = \frac{l}{\sigma S_{min}} = \frac{2\cancel{h}}{\sigma d 3\cancel{h}} = \frac{2}{3\sigma d}$$



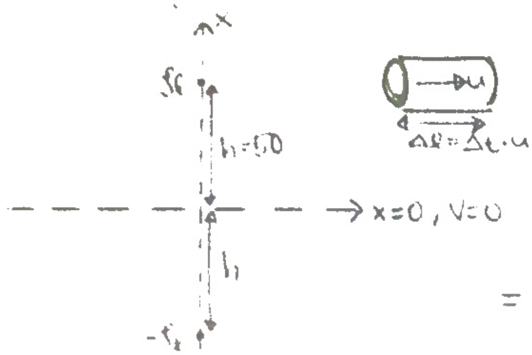
$$\begin{cases} J = \sigma E \\ D = \varepsilon E \end{cases} \qquad \begin{cases} \nabla \cdot D = \rho \\ \nabla \cdot J = -\dfrac{\partial \rho}{\partial t} \end{cases} \xrightarrow[\text{current}]{\text{dc}} \begin{cases} \nabla \cdot D = \rho \\ \nabla \cdot J = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \nabla \cdot (\varepsilon E) = \rho \\ \nabla \cdot (\sigma E) = 0 \end{cases} \xrightarrow[\varepsilon = \sigma \alpha \text{ (const.)}]{\text{assume}} \nabla(\alpha \sigma E) = \alpha \underbrace{\nabla \cdot (\sigma E)}_{=0} = \alpha \cdot 0 = \rho \Rightarrow \rho = 0$$

if $\varepsilon = \alpha \sigma$.

6.2

Dust charged particles are emitting from a chimney $h = 50$m from ground. Wind velocity: 5m/s, they make a horizontal cylindrical charged cloud. (ρ_l)
Current: 100 μA, ground plane is a perfect conductor.
Find E on ground!



E of ρ_l $\rightarrow$ $E = \dfrac{\rho_l}{2\pi\varepsilon_0 r}\hat{r}$

In case of convection current:

$$J = \rho u, \quad i = \frac{\Delta Q}{\Delta t} = \frac{\rho_l \Delta l}{\Delta t} =$$

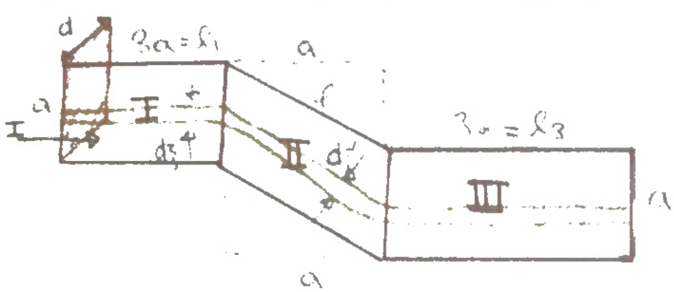
$$= \frac{\rho_l u \cancel{\Delta t}}{\cancel{\Delta t}} = \rho_l u \Rightarrow \rho_l = \frac{i}{u}$$

$$E_x(x=0) = \frac{\rho_l}{2\pi\varepsilon_0 h}(-\hat{x}) + \frac{-\rho_l}{2\pi\varepsilon_0 h}(\hat{x}) = \frac{-\rho_l}{\pi\varepsilon_0 h} = \frac{-i}{\pi\varepsilon_0 h u}\hat{x}$$

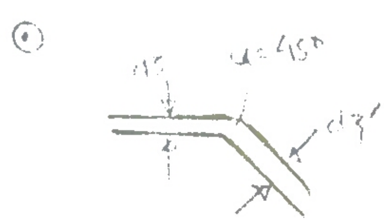
$$E_x(x=0) = \frac{-100 \cdot 10^{-6}}{\pi \cdot 8{,}85 \cdot 10^{-12} \cdot 50 \cdot 5} = 14 \left(\frac{kV}{m}\right)$$

6.11

Resistansberäkning direkt
use two approximation methods to find a lower and upper limit for the resistance between 2 electrodes.



R_{min}?, R_{max}?



$$d_3' = d_3 \sin\alpha = d_3 \frac{1}{\sqrt{2}}$$

Upper bound: use non-physical current tubes $\rightarrow R_{max}$

Storgruppsövning 19/11-13

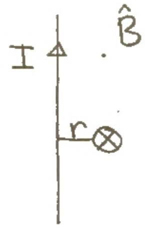
Static magnetic fields, kap 6.

$$\begin{cases} F_e = qE, & \text{electric force on } q \text{ in } E \\ F_m = q\underbrace{V}_{\text{velocity}} \times \underbrace{B}_{\text{magnetic flux density}}, & \text{magnetic force on moving charge } q \text{ in } B \end{cases}$$

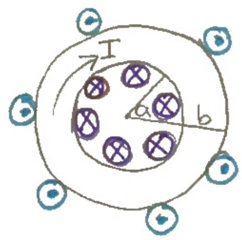
$F = F_e + F_m = q(E + V \times B)$ total electromagnetic force.

Fundamental postulates of magnetostatic (free space):

$$\begin{cases} \nabla \cdot B = 0 \Longrightarrow \oint_S B\,ds = 0 & \text{law of conservation of magnetic flux} \\ \nabla \times B = \mu_0 J \Longrightarrow \oint_C B\,dl = \mu_0 I & \text{Ampere's law} \end{cases}$$



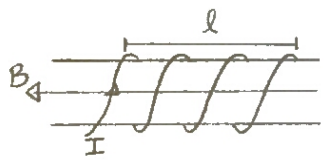
$\hat{B} = \hat{\Phi}\,\dfrac{\mu_0 I}{2\pi r}$



$B = \hat{\Phi}\,\dfrac{\mu_0 N I}{2\pi r}$

$a < r < b$

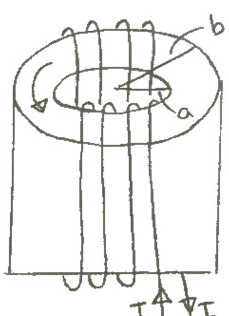
a toroidal-coil

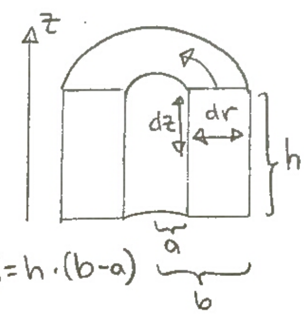


$B = \mu_0 n I$, $n = \dfrac{N}{l}$ (number of turns per length)

P 6.14

A circular toroid with rectangular cross section. Find the total magnetic flux through its cross section.





$S = h \cdot (b-a)$

$\Phi = ? = \int_S B \cdot dS$

$\int B_\phi \cdot dl = \mu_0 N I$

$\Longrightarrow B_\phi = \dfrac{\mu_0 N I}{2\pi r}$, $a < r < b$

$\Phi = \int B_\phi \cdot \underbrace{dS}_{dz\,dr} = (*)$

$$(*) = \frac{\mu_0 N I}{2\pi} \int_0^h \int_a^b \frac{1}{r}\,dz\,dr = \frac{\mu_0 N I h}{2\pi} \ln\left(\frac{b}{a}\right)$$

for its.

Find the percentage of error if: the flux is found by multiplying the $\underbrace{\text{cross-section}}_{S}$ area by flux density at the $\underbrace{\text{mean radius.}}$

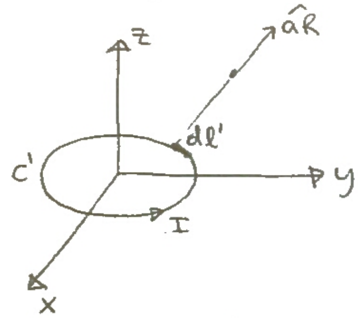
$$B_\phi\left(r=\frac{a+b}{2}\right)=\frac{\mu_0 NI}{2\pi\left(\frac{a+b}{\cancel{2}}\right)}=\frac{\mu_0 NI}{\pi(a+b)}$$

$$\Longrightarrow \phi' = B_\phi\left(r=\frac{a+b}{2}\right)\cdot h(b-a) \Longrightarrow \phi' = \frac{\mu_0 NIh}{\pi}\left(\frac{b-a}{b+a}\right)$$

$$\%\ \text{error} = \frac{\phi'-\phi}{\phi} = \left[\frac{2(b-a)}{(b+a)\ln(b/a)}-1\right]\cdot 100$$

The Biot-Savart law:

Gives the magnetic field of a current carrying circuit.



Vector magnetic potential (A)

$$B=\nabla\times A \xrightarrow[\substack{\text{assume}\\ \nabla\cdot A=0}]{} \nabla^2 A = -\mu_0 J \Longrightarrow A=\frac{\mu_0}{4\pi}\int_{V'}\frac{J}{R}\,dV'$$

For a closed circuit $A=\frac{\mu_0 I}{4\pi}\oint_{C'}\frac{dl'}{R} \Longrightarrow B=\frac{\mu_0 I}{4\pi}\int\frac{dl'\times a_R}{R^2}$

$B=\oint_{C'} dB \quad , \ dB=\frac{\mu_0 I}{4\pi}\left(\frac{dl'\times \mathbb{R}}{R^3}\right)$

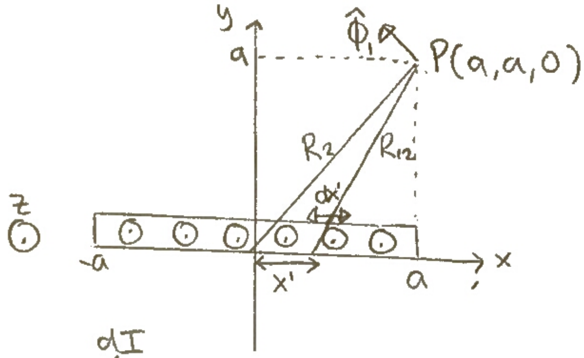
unit vector from source point to the field point

7.2

Biot-Savart law, a very long flat metal strip of width 2a, located in x-z-plane. The current is distributed uniformly. Find the magnitude and direction of B at point P.

$B(P)=?$

The current is dist. uniformly $\Rightarrow I=I_0\hat{z} \Longrightarrow \left(J=\frac{I_0}{2a}\hat{z}\right)$



$$dB = \frac{\mu_0 \overbrace{J_s dx'}^{dI}}{2\pi R_{12}} \hat{\Phi}_1$$

$$\begin{cases} \mathbb{R}_{12} = \mathbb{R}_2 - \mathbb{R}_1 = \hat{x}a + \hat{y}a - x'\hat{x} = \hat{x}(a-x') + \hat{y}a \\ R_{12} = \sqrt{(a-x')^2 + a^2} \end{cases}$$

$$\hat{\Phi}_1 = \frac{d\ell' \times \mathbb{R}_{12}}{R_{12}} = \frac{\hat{z} \times \mathbb{R}_{12}}{R_{12}} = \frac{\hat{y}(a-x') - \hat{x}(a)}{\sqrt{(a-x)^2 + a^2}}$$

$$dB = \frac{\mu_0 (I_0/2a)}{2\pi} \frac{-\hat{x}a + \hat{y}(a-x')}{(a-x')^2 + a^2} dx' \Longrightarrow B = \int_{x'=-a}^{a} dB$$

$$\Longrightarrow dB = \hat{x}\, dB_x + \hat{y}\, dB_y \Longrightarrow B_x = \int_{x'=-a}^{a} \frac{-\mu_0 I_0}{4\pi} \frac{dx'}{(a-x')^2 + a^2} =$$

$$= \left\{ \begin{array}{l} a - x' = \xi,\ d\xi = -dx' \\ x' = -a \Longrightarrow \xi = 2a \\ x' = a \Longrightarrow \xi = 0 \end{array} \right\} = B_x = \int_{\xi=2a}^{0} \frac{\mu_0 I_0}{4\pi a} \frac{d\xi}{\xi^2 + a^2} =$$

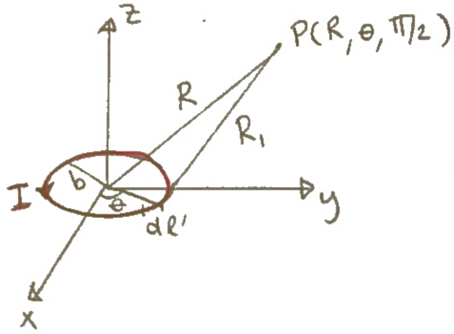
$$= \frac{\mu_0 I_0}{4\pi} \left[\frac{1}{a} \arctan\left(\frac{\xi}{a}\right) \right]_{2a}^{0} = \boxed{\frac{-\mu_0 I_0}{4\pi a} \arctan(2)}$$

$$B_y = \frac{\mu_0 I_0}{4\pi a} \int_{x'=-a}^{a} \frac{(a-x')}{(a-x')^2 + a^2} dx' = \{p.s.s\} = \frac{-\mu_0 I_0}{4\pi a} \int_{2a}^{a} \frac{\xi}{\xi^2 + a^2} d\xi =$$

$$= \frac{-\mu_0 I_0}{4\pi a} \left[\frac{1}{2} \ln(\xi^2 + a^2) \right]_{2a}^{0} = \frac{\mu_0 I_0}{8\pi a} \ln\left(\frac{4a^2 + a^2}{a^2}\right) \boxed{\frac{\mu_0 I_0}{8\pi a} \ln(5)}$$

$$\Longrightarrow B(P) = B_x + B_y = \frac{\mu_0 I_0}{4\pi a} \left(-\hat{x} \arctan(2) + \hat{y} \frac{\ln(5)}{2} \right)$$

The magnetic dipole: a small circular loop carries a current I.



magnetic dipole moment

$$A = \mu_0 \frac{\mathbf{m} \times \hat{a}_R}{4\pi R^2}, \quad \mathbf{m} = \hat{a}_z \pi b^2 I = \hat{a}_z S I = \hat{a}_z m$$

$$\mathbf{B} = \frac{\mu_0 m}{4\pi R^3}\left(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta\right)$$

7.10

A magnetic dipole $\mathbf{m} = \hat{z} m$ is in origin. Find the $\underbrace{\text{magnetic flux}}_{\Phi}$ through the ring.

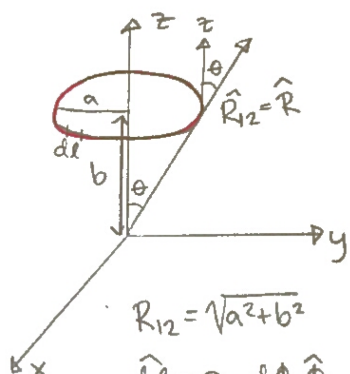
$$\Phi = \int_S B \cdot dS = ? = \int_S (\nabla \times A) \cdot dS = \oint_C A \cdot dl$$

$$A = \mu_0 \frac{\mathbf{m} \times \mathbf{R}_{12}}{4\pi R_{12}^2} = \frac{\mu_0 \hat{z} m \times \hat{R}}{4\pi (a^2+b^2)} =$$

$$= \frac{\mu_0}{4\pi(a^2+b^2)} m \underbrace{\sin\theta\, \hat{\phi}}_{\hat{z} \times \hat{R}}$$

$$\Phi = \oint_C A\, dl = \int_{\phi=0}^{2\pi} \frac{\mu_0 m}{4\pi(a^2+b^2)} \cdot \frac{a}{\sqrt{a^2+b^2}} \hat{\phi} (\hat{\phi} a\, d\phi) =$$

$$= \frac{\mu_0 m a^2}{2(a^2+b^2)^{3/2}}$$



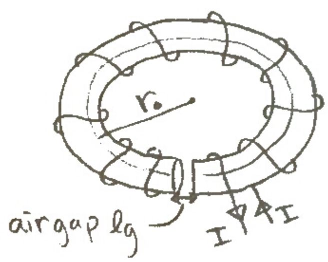
$R_{12} = \sqrt{a^2+b^2}$

$\hat{dl} = a \cdot d\phi\, \hat{\phi}$

$\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$

ex 6.10 i boken

Magnetic circuits, N-turns of wire around ferromatic Toroidal-coil. Find B and H both in core and in air-gap.



Permeability: μ

mean radius: r_0

cross-section of coil: $a << r_0$

forts →

Neglecting fringing and leakage: $\mathbb{B}_f = \mathbb{B}_g = \hat{\Phi} B_f$

$$\longrightarrow \begin{cases} \mathbb{H}_f = \hat{\Phi}\,\dfrac{B_f}{\mu} & \text{magnetic flux intensity in coil} \\ \mathbb{H}_g = \hat{\Phi}\,\dfrac{B_f}{\mu_0} & \text{magnetic flux intensity in airgap} \end{cases}$$

Ampere's law:

$$\oint \mathbb{H}\cdot dl = NI_0 \longrightarrow \frac{B_f}{\mu}(2\pi r_0 - l_g) + \frac{B_f}{\mu_0}\cdot l_g = NI_0$$

$$\longrightarrow B_f = \hat{\Phi}\,\frac{\mu\mu_0 N I_0}{\mu_0(2\pi r_0 - l_g) + \mu l_g}$$

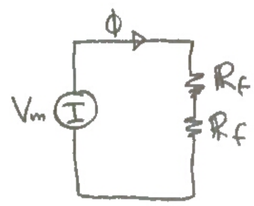
Assume B constant $\Longrightarrow$ $\Phi = BS = \dfrac{NI_0}{\frac{2\pi r_0 - l_g}{S\cdot} + \frac{l_g}{S\mu_0}}$

magneto motive force.

$$\phi = \frac{V_m}{R_f + R_g}$$ — reluctance

$R_f = \dfrac{2\pi r_0 - l_g}{\mu S}$, $R_g = \dfrac{l_g}{S\mu_0}$

$I = \dfrac{V_e}{R_f + R_g}$

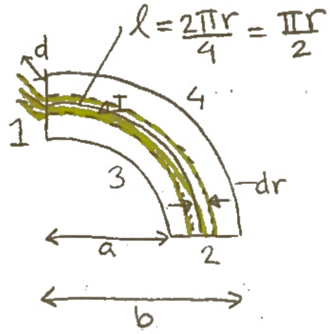


Storgruppsövning 19/11-13 em.

Resistansberäkningar, övre och undre gräns

6.16

quarter disk, $\left(\begin{array}{l} \sigma = \text{conductivity} \\ d = \text{thickness} \end{array} \right)$



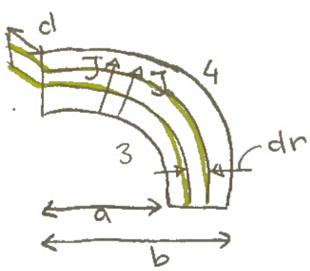
a) Calculate R_{12} (between electrodes 1 & 2)

$$dG = \frac{\sigma S}{\ell} = \frac{\sigma(dr \cdot d)}{\pi r/2} = \frac{2\sigma d}{\pi r} dr$$

$$G_{12} = \int_{r=a}^{b} dG = \int_a^b \frac{2\sigma d}{\pi r} dr = \frac{2\sigma d}{\pi} \ln\left(\frac{b}{a}\right)$$

$$\Longrightarrow R_{12} = \frac{1}{G_{12}} = \frac{\pi}{2\sigma d \ln(b/a)}$$ parallellkoppling

b)



$$dR = \frac{\ell}{\sigma S} = \frac{dr}{\sigma \cdot \pi r d/2} = \frac{2}{\sigma \pi r d} dr$$

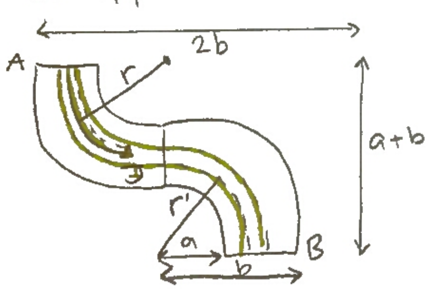
$$S = \frac{\pi}{2} r d$$

$$R_{34} = \int_{r=a}^{b} dR = \int_a^b \frac{2}{\sigma \pi r d} dr = \frac{2}{\sigma \pi d} \ln(b/a)$$ seriekoppling

$$R_{12} \cdot R_{34} = \frac{\pi}{2\sigma d \ln(b/a)} \cdot \frac{2 \ln(b/a)}{\sigma \pi d} = \left(\frac{1}{\sigma d}\right)^2$$

6.17

A thin plate, has thickness $d = 0{,}1\,\text{mm}$ and conductivity σ.
Find upper and lower bounds for R_{AB} if $2a = b$.



- Upper bound: assume current tubes.

$$dG = \frac{\sigma S}{\ell}, \quad \begin{cases} \ell = \frac{\pi}{2} r + \frac{\pi}{2} \underbrace{(a+b-r)}_{r'} \\ S = d \cdot dr \end{cases}$$

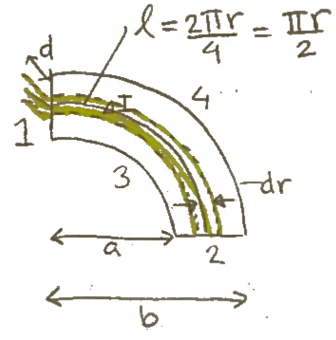
forts. →

Storgruppsövning 19/11-13 em.

Resistansberäkningar, övre och undre gräns

6.16

quarter disk, $\begin{pmatrix} \sigma = \text{conductivity} \\ d = \text{thickness} \end{pmatrix}$



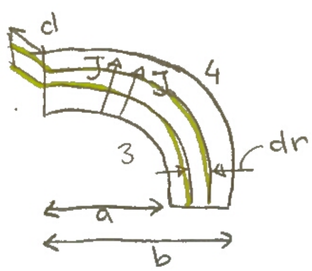
a) Calculate R_{12} (between electrodes 1 & 2)

$$dG = \frac{\sigma S}{l} = \frac{\sigma(dr \cdot d)}{\pi r/2} = \frac{2\sigma d}{\pi r}\,dr$$

$$G_{12} = \int_{r=a}^{b} dG = \int_a^b \frac{2\sigma d}{\pi r}\,dr = \frac{2\sigma d}{\pi} \ln\left(\frac{b}{a}\right)$$

$$\Longrightarrow R_{12} = \frac{1}{G_{12}} = \frac{\pi}{2\sigma d \ln(b/a)}$$ parallellkoppling

b)



$$dR = \frac{l}{\sigma S} = \frac{dr}{\sigma \cdot \pi r d/2} = \frac{2}{\sigma \pi r d}\,dr$$

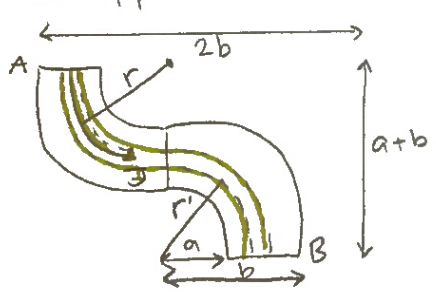
$$S = \frac{\pi}{2} r d$$

$$R_{34} = \int_{r=a}^{b} dR = \int_a^b \frac{2}{\sigma \pi r d}\,dr = \frac{2}{\sigma \pi d} \ln(b/a)$$ seriekoppling

$$R_{12} \cdot R_{34} = \frac{\pi}{2\sigma d \ln(b/a)} \cdot \frac{2 \ln(b/a)}{\sigma \pi d} = \left(\frac{1}{\sigma d}\right)^2$$

6.17

A thin plate, has thickness $d = 0{,}1\,\text{mm}$ and conductivity σ.
Find upper and lower bounds for R_{AB} if $2a = b$.



- Upper bound: assume current tubes.

$$dG = \frac{\sigma S}{l}, \quad \begin{cases} l = \frac{\pi}{2} r + \frac{\pi}{2}(\underbrace{a+b-r}_{r'}) \\ S = d \cdot dr \end{cases}$$

forts. →

$\longrightarrow dG = \frac{\sigma d\, 2}{\pi(a+b)}\, dr$

$\Longrightarrow G = \int_{r=a}^{b} dG = \int_a^b \frac{2\sigma d}{\pi(a+b)}\, dr = \frac{2\sigma d(b-a)}{\pi(a+b)} = \{b=2a\} = \frac{2\sigma d}{3}$

$\Longrightarrow R = \frac{3\pi}{2\sigma d} = 0,471\ \Omega$

lower bound: assume an equipotential surface as real-line

$R^{\ell} = 2R_{12} = \cancel{2}\cdot\frac{\pi}{\cancel{2}\sigma d\ln(b/a)} = \frac{\pi}{\sigma d\ln(2)} = 0,453\ \Omega$

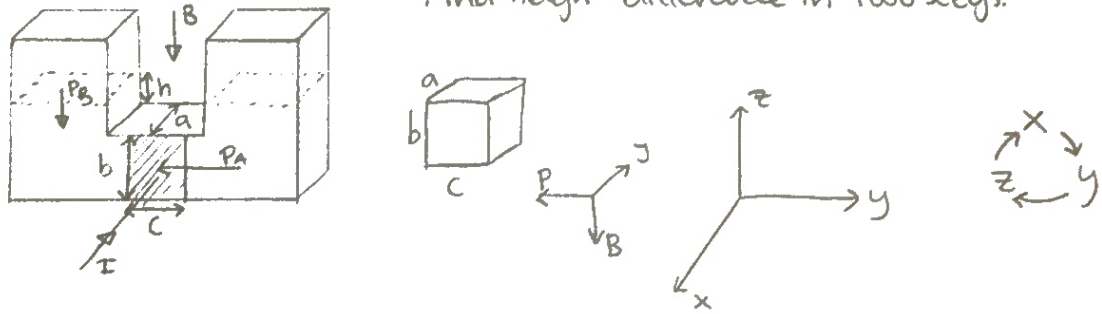
$\uparrow$ from 6.16

$\Longrightarrow$ $0,453 <$ actual $R_{AB} < 0,471$

7.8

(u-tube) in the figure is filled with a conductor fluid, mass density η.

Find height difference in two legs.



$J_0 = \frac{J}{bc}$

$\begin{cases} \mathbf{J} = -\hat{x}J_0 \\ \mathbf{B} = -B_0\hat{z} \end{cases}$

$F_m = q\mathbf{v}\times\mathbf{B} = \rho\Delta V\,\mathbf{v}\times\mathbf{B} \Longrightarrow \frac{\mathbf{F}}{\Delta V} = \rho\,\mathbf{v}\times\mathbf{B}$

$\rho = q/\Delta V \Longrightarrow q = \rho\Delta V$

$\mathbf{J} = \rho\mathbf{v}$ convection current

$\Longrightarrow \frac{\mathbf{F}}{\Delta V} = \mathbf{J}\times\mathbf{B} = -J_0B_0\hat{y}\ \left(\frac{N}{m^3}\right) \quad \left((-\hat{x}J_0)\times(-B_0\hat{z})\right)$

$F_A = \frac{F}{\Delta V}\cdot abc = J_0B_0abc$ (the force in cubic region)

$P_A = \frac{F_A}{ab}$ (the force acts on the shaded area, cause the pressure P_A)

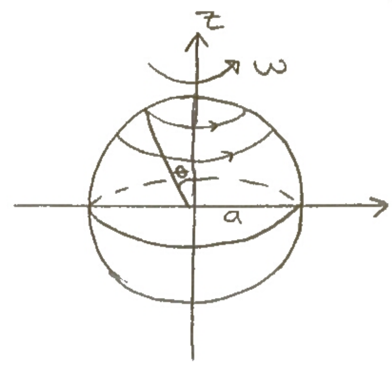
$P_A = \frac{J_0B_0abc}{ab} = J_0B_0c$

$$P_B = \eta g h = J_0 B_0 C \Longrightarrow h = \frac{J_0 B_0 C}{\eta g}$$

mass density

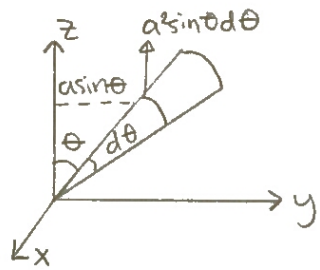
7.7

Biot-Savart law, a metal sphere of radius a with charge Q, distributed uniformly
- angular velocity ω
- Find B in center!



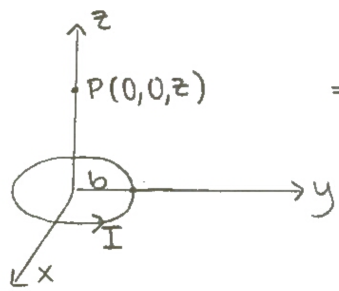
$$\omega = \frac{d\phi}{dt}, \quad \rho_s = \frac{Q}{4\pi a^2}$$

$$dq = \rho_s\, ds = \rho_s\, ds_R = \rho_s a^2 \sin\theta\, d\theta\, d\phi$$



$$di = \frac{dq}{dt} = \rho_s a^2 \sin\theta\, d\theta \frac{d\phi}{dt} = $$

$$= \rho_s a^2 \sin\theta\, \omega\, d\theta$$



$$\Longrightarrow \mathbb{B}(P) = \hat{z} \frac{\mu_0 I b^2}{2(z^2+b^2)^{3/2}} \quad \left\{ \begin{array}{l} I = di \\ b = a\sin\theta \\ z = a\cos\theta \\ z^2+b^2 = a^2 \end{array} \right\}$$

$$\Longrightarrow d\mathbb{B} = \hat{z} \frac{\mu_0 di (a\sin\theta)^2}{2a^3} =$$

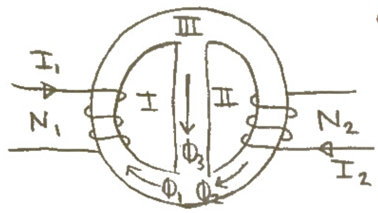
$$= \hat{z} \frac{\mu_0 Q \omega}{8\pi a} \sin^3\theta\, d\theta$$

$$\mathbb{B} = \int_{\theta=0}^{\pi} d\mathbb{B} = \hat{z} \frac{\mu_0 Q \omega}{6\pi a}$$

8.5

> **Rules:** $\sum_j N_j I_j = \sum R_k \Phi_k$ around a close path in a magnetic circuit.
>
> $\sum_j \Phi_j = 0$ in a junction

Iron ring $(a = 7.5\,cm)$ and $(A_1 = 1.2\,cm^2)$
$(A_3 = 0.8\,cm^2)$ $\left(\begin{matrix} N_1 = 160 \text{ turns} \\ N_2 = 120 \text{ turns}\end{matrix}\right)$ $I_1 = I_2 = 2\,mA$



a) Calculate the flux through the bridge (Φ_3)

$$\begin{cases} \mathrm{I}: R_1 \Phi_1 + R_3 \Phi_3 = N_1 I_1 \\ \mathrm{II}: R_2 \Phi_2 - R_3 \Phi_3 = N_2 I_2 \\ \mathrm{III}: \Phi_1 = \Phi_2 + \Phi_3 \end{cases}$$

$$R_1 = R_2 = \frac{l_1}{\mu A_1} = \frac{\pi a}{\mu A_1}$$

$$R_3 = \frac{l_3}{\mu A_3} = \frac{2a}{\mu A_3}$$

In matrix-form:

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_1 + R_3 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} N_1 I_1 \\ N_2 I_2 \end{bmatrix} \Longrightarrow \Phi_3 = 2{,}639 \cdot 10^{-9} \ (wb)$$

b) we assume $I_2 = 0$, Find I_1 in order to have $\Phi_3 = 60 \cdot 10^{-6}\,wb$
use magnetication chart for iron

$$\Phi_3 = 60 \cdot 10^{-6} \Longrightarrow \Phi_3 = B_3 A_3 \Longrightarrow B_3 = \frac{\Phi_3}{A_3} = \frac{60 \cdot 10^{-6}}{0{,}8 \cdot 10^{-4}} = 0.75 \ (T)$$

use chart for iron: $H_3 = 4200 \ (A/m)$

$$\oint H \cdot dl = I \xrightarrow[\text{in loop II}]{} H_3 \cdot 2a - H_2 \cdot \pi a = 0 \Longrightarrow H_2 = 2800 \ (A/m)$$

use chart $\Longrightarrow B_2 = 0.65 \ (T)$, $B_2 = \Phi_2 / A_2 \Longrightarrow \Phi_2 = 78 \cdot 10^{-6}\,wb$

in junction III: $\Phi_1 = \Phi_2 + \Phi_3 \Longrightarrow \Phi_1 = 138 \cdot 10^{-6}\,wb$

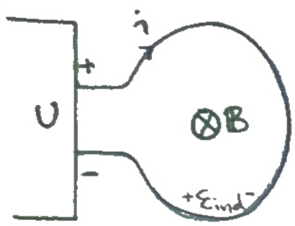
$\Longrightarrow B_1 = \frac{\Phi_1}{A_1} = 1{,}15 \ (T)$, chart $\Longrightarrow H_1 = 20000 \ (A/m)$

forts. →

loop I $\Longrightarrow \oint H \cdot dl = I \Longrightarrow H_1 \cdot \pi a + H_3 \cdot 2a = NI_1$

$\Longrightarrow I_1 = \dfrac{H_1 \cdot \pi a + H_3 \cdot 2a}{N} = 33{,}6\ A$

Föreläsning 20/11-13



$$\mathcal{E}_{ind} = -\ell \frac{di}{dt} = -\frac{d\Phi}{dt}$$

Beräkning av induktans:

1) Antag i_1
2) Beräkna B_1
3) Beräkna Φ_{12}
4) Beräkna Λ_{12}
5) Bilda $\ell_{12} = \Lambda_{12}/i_1$

Neumans formel:

(1) (2) $\ell_{12} = \ell_{21}$

Magnetisk energi 6.12

Det kostar energi att bygga upp B-fältet.

Om slingan resistanslös: $U + \mathcal{E}_{ind} = 0$

Hur mycket kostar det? : dW_m (ändring i B-fältets energi) $= dW_{batteri} =$
$= U \cdot i \cdot dt = -\mathcal{E}_{ind}\, i\, dt = i\, d\Phi$

För ensam slinga:

Enligt def: $\Phi = \ell i \Longrightarrow dW_m = i \ell di$

Integrera: $W_m = \int_0^i i \ell di = \frac{1}{2}\ell i^2$

För N st slingor:

Arbete i slinga k: $dW_k = i_k \cdot d\Phi_k$

För alla slingor: $dW_m = \sum_{k=1}^{N} dW_k = \sum_{k=1}^{N} i_k d\Phi_k$

Låt $i_k = \alpha I_k$

$\Phi_k = \alpha \bar{\Phi}_k$

$\frac{d\Phi_k}{d\alpha} = \bar{\Phi}_k$

Totalt arbete: $W_m = \int dW_m = \sum_{k=1}^{N} I_k \bar{\Phi}_k \int_0^1 \alpha\, d\alpha = \frac{1}{2}\sum_{k=1}^{N} I_k \bar{\Phi}_k$

Magnetisk energi i $\mathbb{J}$ och $\mathbb{A}$

Med $\Phi = \int \mathbb{B} \, d\mathbb{S} = \oint \mathbb{A} \cdot d\mathbb{l}$

$I = \int \mathbb{J} \, d\mathbb{s}$

$$\Rightarrow W_m = \frac{1}{2}\int_{V'} \mathbb{J} \cdot \mathbb{A} \, dV'$$

I $\mathbb{H}$ och $\mathbb{B}$

$W_m = \frac{1}{2}\int_{V'} \mathbb{J} \cdot \mathbb{A} \, dV' = \left\{ \begin{array}{l} \text{postulat vektoridentitet} \\ \text{div. teorem} \end{array} \right\} = \dots =$

$= \frac{1}{2}\int_{V'} \mathbb{H} \cdot \mathbb{B} \, dV'$

Energimetod för kraftberäkningar

I en magnetisk krets definierar I_k och Φ_k energin.

1) I_k konstant: Φ_k ändras

2) Φ_k konstant: I_k ändras.

1) $W_m = \frac{1}{2} \sum_{k=1}^{N} I_k \Phi_k$ Initial energi

$W_m' = \frac{1}{2} \sum_{k=1}^{N} I_k (\Phi_k + \partial \Phi_k)$ slutlig energi

Ändring: $dW_m = \frac{1}{2} \sum_{k=1}^{N} I_k \partial \Phi_k$

Batteriet tillför: $dW_s = \sum_{k=1}^{N} I_k \partial \Phi_k$ (Måste gå jämnt ut)

Mek. energi: $dW_{mek} = \mathbb{F}_I \cdot d\mathbb{l}$

Energiprincipen: $dW_s = dW_{mek} + dW_m$

$\mathbb{F}_I \cdot d\mathbb{l} = dW_m = (\nabla W_m) \cdot d\mathbb{l}$ $\quad (\mathbb{F}_I = \nabla W_m)$

2) $\mathbb{F}_\Phi = -\nabla W_m$ (analogt resonemang)

Repetition inför duggan

E-statik

Postulat: $\begin{cases} \nabla \cdot \mathbb{E} = \rho/\varepsilon_0 \\ \nabla \times \mathbb{E} = 0 \end{cases}$

Def. av kraft: $\mathbb{F} = q\,\mathbb{E}$

Gauss lag: $\int_S \mathbb{E}\, d\mathbb{S} = \frac{Q}{\varepsilon_0}$ kräver symmetri!

Pkt-laddning: $E(R) = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} \hat{\mathbb{R}}$

Superposition: $E(R) = \frac{1}{4\pi\varepsilon_0} \int \frac{dQ}{R^2} \hat{\mathbb{R}} = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho\, dV'}{R^2} \hat{\mathbb{R}}$

Potential: $\nabla \times \mathbb{E} = 0 \implies \mathbb{E} = -\nabla V$

$V(R) = \int_R^{R_{Ref}} \mathbb{E} \cdot d\mathbb{l} + V(Ref)$

Pkt-laddning: $V(R) = \frac{1}{4\pi\varepsilon_0} \frac{q}{R}$

Integrera för godt. laddningsförd.: $V(R) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(R)\, dV'}{R}$

Metall: $\mathbb{E} = 0 \implies V =$ konstant (metaller är ekvipot.ytor)

Materialmodell:

$\mathbb{D} = \varepsilon_0 \mathbb{E} + \mathbb{P} = \varepsilon_0 \varepsilon_r \mathbb{E}$

Randvillkor: $E_{1t} = E_{2t}$ (e-fältets tang. komp. alltid kont.)

$D_{1n} - D_{2n} = \rho_s$ (D-fältets norm.komp.)

Kapacitans: $C = Q/\Delta V$, hur mkt energi som kan lagras i ett system

Energi: $W_e = \frac{1}{2} \int_{V'} V(R)\, \rho(R)\, dV' = \frac{1}{2} \int_{V'} \mathbb{D} \cdot \mathbb{E}\, dV'$ tänk på vilket område som ska integreras över!

Kraft: $\mathbb{F}_Q = -\nabla W_e$

$\mathbb{F}_V = \nabla W_e$

Coulombs lag: $\mathbb{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{R_{12}^2} \hat{\mathbb{R}}_{12}$

Spegling:

Ström: $Di = \frac{DQ}{Dt}$

Kontinuitetsekv.: $\int \mathbb{J} d\mathbb{S} = \frac{\partial Q}{\partial t}$ (laddningar kan ej förstöras, men flyttas.)

$\nabla \cdot \mathbb{J} = 0 \Leftarrow$ vid likström!

Ohms lag: $\mathbb{J} = \sigma \mathbb{E}$

Joules lag: $\underset{\text{(effekt)}}{P} = \int_{V'} \mathbb{E} \cdot \mathbb{J} dV'$

Randvillkor: $J_{1n} = J_{2n}$ (norm. komp. kont.)

$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}$ (tang. komp. beror på materialen)

Resistans: $R = \frac{\Delta V}{I}$ resistans- och kapacitansberäkningar är kopplade till varandra.

Approx. beräkning: Strömrör $\Longrightarrow$ för hög resistans

Ekv. potytor $\Longrightarrow$ för låg resistans.

<u>Magnetostatik</u>

Postulat: $\begin{cases} \nabla \cdot \mathbb{B} = 0 \\ \nabla \times \mathbb{B} = \mu_0 \mathbb{J} \end{cases}$

Kraft: $\mathbb{F} = q(\mathbb{u} \times \mathbb{B})$

Amperes lag: $\int \mathbb{B} \cdot d\mathbb{l} = \mu_0 I_0$

"linjeström": $B = \frac{\mu_0 I}{2\pi R}$

Biot-Savart: $\mathbb{B} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbb{J}(\mathbb{R}') \times \hat{\mathbb{R}}}{R^2} dV'$

Vektorpotential: $\nabla\cdot\mathbb{B}=0 \Longrightarrow \mathbb{B}=\nabla\times\mathbb{A}$

↑ vår vektor-potential

$$\mathbb{A}=\frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbb{J}(\mathbb{R}')}{R}\,dV'$$

Materialmodell:

$$\mathbb{H}=\frac{\mathbb{B}}{\mu_0}-\mathbb{M}$$

$$\Longrightarrow \mathbb{B}=\mu_0\mu_r\mathbb{H}$$

Randvillkor: $B_{1n}=B_{2n}$

$$(\mathbb{H}_1-\mathbb{H}_2)_{tang}=\mathbb{J}_g\times\hat{n}_2$$

Energi: $W_m=\frac{1}{2}\int_{V'}\mathbb{J}\,\mathbb{A}\,dV'=\frac{1}{2}\int_{V'}\mathbb{A}\cdot\mathbb{B}\,dV'$

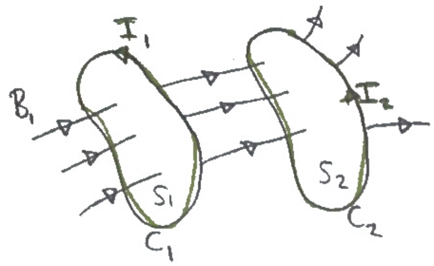
Kraft: $\mathbb{F}_I=\nabla W_m$

$\mathbb{F}_\Phi=-\nabla W_m$

Amperes kraftlag: $\mathbb{F}_m=\int_{V'}\mathbb{J}\times\mathbb{B}\,dV'$

Storgruppsövning 20/11-13

Self inductance and mutual inductance



Current I_1 in $C_1 \Rightarrow B_1 \Rightarrow$ pass through S_2

from Biot-Savart's law:

$B_1 \propto I_1 \Rightarrow \phi_{12} = \int_{S_2} B_1 \, ds \Rightarrow \phi_{12} = \ell_{12} I_1$

$\begin{cases} \ell_{12}: \text{mutual inductance between } C_1 \,\&\, C_2. \\ \ell_{11}: \text{self inductance of loop } C_1. \end{cases}$

flux linkage

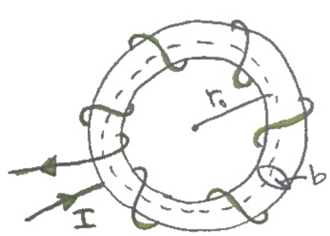
$\ell_{12} = \Lambda_{12}/I_1$

$\ell_{11} = \Lambda_{11}/I_1$

$\begin{cases} N_1 \text{ turns in } C_1 \Rightarrow \Lambda_{11} = N_1 \phi_{11} \\ N_2 \text{ turns in } C_2 \Rightarrow \Lambda_{12} = N_2 \phi_{12} \end{cases}$

P 6.35

Find the self-inductance of a toroidal coil, N-turns of wire, mean-radius = r_0, circular cross-section with $r=b$, $b \ll r_0$.



Use Ampere's law: $\oint_\ell B \cdot d\ell = \mu_0 I_{in}$

$\Rightarrow B_\phi 2\pi r = \mu_0 N I \Rightarrow B_\phi = \frac{\mu_0 N I}{2\pi r_0}$

$\phi = \int_S B \cdot ds = B_\phi \cdot S$ (cause $b \ll r_0$)

$\phi = B_\phi \pi b^2 \longrightarrow \phi = \frac{\mu_0 N I}{2\cancel{\pi} r_0} \cdot \cancel{\pi} b^2 = \frac{\mu_0 N I b^2}{2 r_0}$

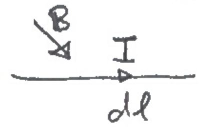
$\ell_{11} = \frac{\Lambda_{11}}{I_1} = \frac{N\phi}{I_1} = \frac{\mu_0 N^2 \cancel{I} b^2}{\cancel{I} 2 r_0} = \frac{\mu_0 N^2 b}{2 r_0}$

To find a self-inductance:

1) choose an appropriate coordinate system
2) assume a current
3) find B by Biot-Savart or by Ampere's law. symmetry
4) find the flux $\phi = \int_S B \cdot ds$
5) find flux linkage ($\Lambda = N\phi$)
6) find $\ell = \Lambda / I$

Magnetic force on a current-carrying conductor

$F_m = I\, dl \times B$ (N)



Magnetic force on a closed circuit with current I, in magnetic field B.

$$F_m = I \oint_C dl \times B \quad (N)$$

When we have 2 circuit carrying I_1 & I_2:
the force F_{21} on circuit C_1:

$$F_{21} = I_1 \oint_{C_1} dl_1 \times B_{21}$$

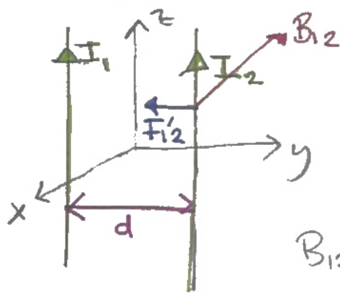
caused by current I_2 in C_2.

By Biot-Savart law: $B_{21} = \frac{\mu_0 I}{4\pi} \oint_{C_2} \frac{dl_2 \times \hat{a}_{R21}}{R_{21}^2}$

$$F_{21} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{dl_1 \times (dl_2 \times \hat{a}_{R21})}{R_{21}^2}$$

Ampere's law of force between two current-carrying circuit.

example
force per unit length of two long parallell wire I_1 & I_2.



$F'_{12} = I_2 (\hat{a}_z \times B_{12})$ force per unit length on wire 2.

magnetic field at wire 2 from current I_1.

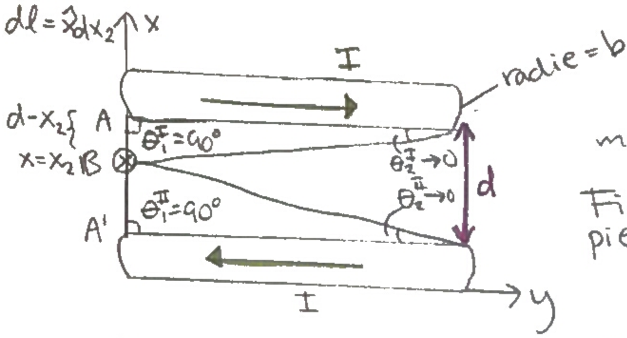
$B_{12} = -\hat{a}_x \frac{\mu_0 I_1}{2\pi d} \Rightarrow$ substitute in F'_{12}.

$$F'_{12} = -\hat{a}_y \frac{\mu_0 I_1 I_2}{2\pi d}$$

$$F'_{21} = -F'_{12}.$$

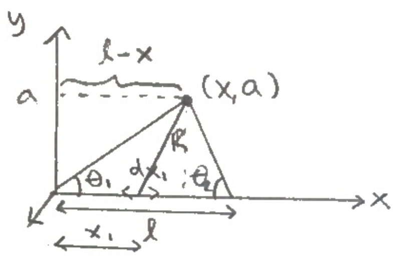
P6.46

The bar AA' in (Fig 6.53) connecting two very long parallell lines with current I.



Find direction and magnitude of F_m on AA' !

First we find B caused by a piece of line current.



$$dB = \frac{\mu_0 I}{4\pi}\left(\frac{dl' \times \mathbb{R}}{R^2}\right)$$

$$\begin{cases} \mathbb{R} = \hat{a}_x (x - x_1) + a\,\hat{a}_y \\ R = \sqrt{(x-x_1)^2 + a^2} \\ dl' = \hat{a}_x\, dx_1 \end{cases}$$

$$\Longrightarrow \quad dl' \times \mathbb{R} = \hat{z}\, a\, dx_1$$

$$d\mathbb{B} = \hat{z}\frac{\mu_0 I}{4\pi}\frac{a}{((x-x_1)^2+a^2)^{3/2}}\, dx_1$$

$$\mathbb{B} = \int_{x_1=0}^{l} d\mathbb{B} = \hat{z}\frac{\mu_0 I a}{4\pi}\int_0^l \frac{dx_1}{((x-x_1)^2+a^2)^{3/2}} = \left\{\begin{aligned} &\int \left[(x-a)^2+b^2\right]^{-3/2} dx = \\ &= \frac{x-a}{b^2\sqrt{(x-a)^2+b^2}}, \quad a = x_1,\ b^2 = a^2 \end{aligned}\right\} =$$

$$= \hat{z}\frac{\mu_0 I \cancel{a}}{4\pi}\left[\frac{1}{a\cancel{^2}}\frac{x_1 - x}{\sqrt{(x-x_1)^2+a^2}}\right]_{x_1=0}^{l} = \hat{z}\frac{\mu_0 I}{4\pi a}\left[\underbrace{\frac{l-x}{\sqrt{(l-x)^2+a^2}}}_{\cos\theta_2} + \underbrace{\frac{x}{\sqrt{x^2+a^2}}}_{\cos\theta_1}\right] =$$

$$= \hat{z}\frac{\mu_0 I}{4\pi a}(\cos\theta_2 + \cos\theta_1)$$

$$\mathbb{B}(x = x_2, y = 0, z = 0) = \frac{\mu_0 I}{4\pi(d-x_2)}(0+1)\hat{z} + \frac{\mu_0 I}{4\pi x_2}(0+1)\hat{z} =$$

$$= \hat{z}\frac{\mu_0 I}{4\pi}\left(\frac{1}{d-x_2} + \frac{1}{x_2}\right)$$

$$F_m = \int_l I\, dl \times \mathbb{B} \quad , \quad F_m = \int_{x_2=b}^{d-b} I(\hat{x}\, dx_2) \times \left(\frac{\mu_0 I}{4\pi}\left(\frac{1}{d-x_2} + \frac{1}{x_2}\right)\right)\hat{z}$$

forts $\rightarrow$

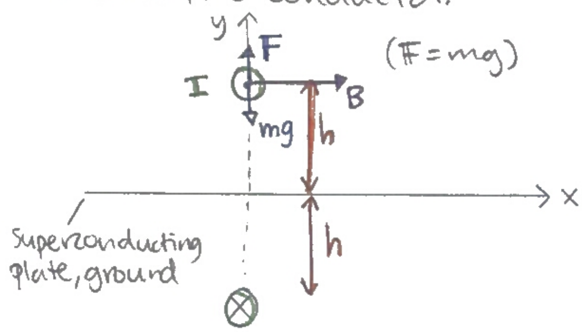
$$\mathbb{F}_m = \hat{y}\int_{x_2=b}^{d-b} -\frac{\mu_0 I^2}{4\pi}\left(\frac{1}{x_2} + \frac{1}{d-x_2}\right)dx_2$$

$$\mathbb{F}_m = -\hat{y}\,\frac{\mu_0 I^2}{4\pi}\Big[\ln(x_2) - \ln(d-x_2)\Big]_b^{d-b} = -\hat{y}\,\frac{\mu_0 I^2}{4\pi}\left[\ln\left(\frac{d-b}{b}\right) - \ln\left(\frac{b}{d-b}\right)\right] =$$

$$= -\hat{y}\,\frac{\mu_0 I^2}{2\pi}\ln\left(\frac{(d-b)}{b}\right)$$

7.21

long linear conductor; circular cross-section ($r=a$) float over a long superconductor plate. Find h and I_{min} to lift the conductor.



hint: use image method.

Find the magnetic flux density due to the image current at ($y=h$)

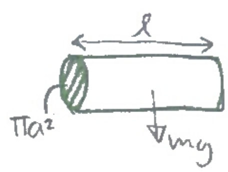
(Assume ℓ length of the wire)

Use Ampere's law : $\mathbb{B} = \frac{\mu_0 I}{2\pi(2h)}\hat{x}$

The force on length ℓ: $\mathbb{F}_m = \int_\ell I\,d\ell \times \mathbb{B} = \int_{z=-\ell/2}^{\ell/2} I(\hat{z}\,dz)\times\left(\frac{\mu_0 I}{4\pi h}\hat{x}\right) =$

$= \frac{\mu_0 I^2}{4\pi h}\,\hat{y}\int_{-\ell/2}^{\ell/2} dz = \frac{\mu_0 I^2 \ell}{4\pi h}\,\hat{y}$ magnetic force at $y=h$

$F_m = mg$



$$\frac{\mu_0 I^2 \ell}{4\pi h} = \eta V g = \eta(\pi a^2 \ell)g$$

$$h = \frac{\mu_0 I^2}{4\pi\eta\pi a^2 g}$$

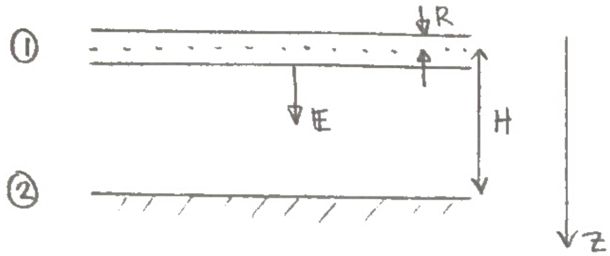
in order to lift the wire, $h \geq a$:

$$\Rightarrow I_{min}^2 = \frac{4\pi a^2 g\eta}{\mu_0}\cdot \underset{a}{\overset{\shortparallel}{h}} \Rightarrow I_{min} = 2\pi a\sqrt{\frac{g\eta a}{\mu_0}}$$

Storgruppsövning 22/11-13

Genomgång av tenta 23/8-13

1.
A phone line is suspended 6 m above the ground. Calculate the capacitance per unit length. (assume the line is very long and has $d = 1\,mm$).



$C = \frac{Q}{\Delta V}$

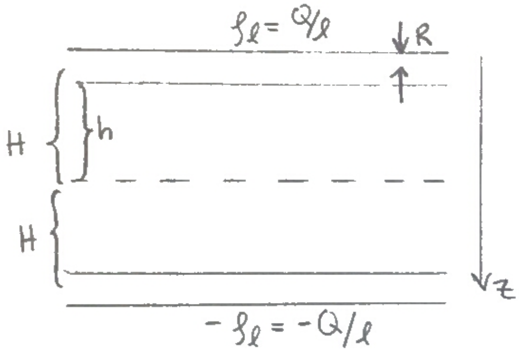
$\rho_\ell = Q/\ell$

$|\Delta V| = \int E\,d\ell$

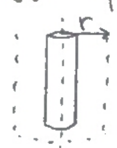
Find E

use image method !!!

use Gauss law to find E



Use Gauss law:



$\oint E\,ds = \frac{Q}{\varepsilon_0}$

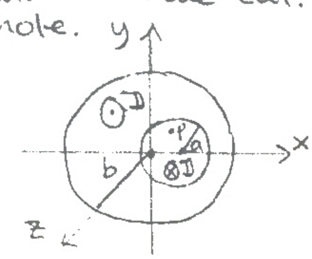
$\Rightarrow E = \frac{Q}{2\pi\varepsilon_0 r \ell} = \frac{\rho_\ell \ell}{\ell\, 2\pi\varepsilon_0 r}$

$E(h) = \frac{\rho_\ell}{2\pi\varepsilon_0}\left(\frac{1}{H-h} + \frac{1}{H+h}\right) \qquad 0 \le h < H-R$

$|\Delta V| = \int_0^{H-R} \frac{Q}{2\pi\varepsilon_0}\left(\frac{1}{H-h} + \frac{1}{H+h}\right)dh = \frac{Q}{2\pi\varepsilon_0}\left(\ln(2H-R) - \ln R\right)$

$C = \frac{Q}{\Delta V} = \frac{Q}{\frac{Q}{2\pi\varepsilon_0}\left(\ln(2H-R) - \ln R\right)} = \frac{2\pi\varepsilon_0}{\ln(2H-R) - \ln R} = 5{,}5\ \frac{pF}{m}$

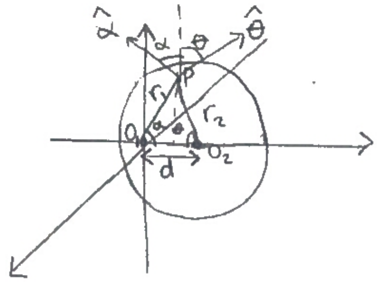
2.
In a very long straight cylinder there is a cylindrical hole-cut. Find the magnitude and direction of $\mathbb{B}$ in the hole. (assume current is uniformly distributed).



The $\mathbb{B}$-field at point P in the cavity is the superposition of 2 $\mathbb{B}$-field:

$\mathbb{B}_1$ = produced by $J = J_0\hat{z}$ in a cylinder with $r = b$

$\mathbb{B}_2$ = ―― " ―― $J = -J_0\hat{z}$ ―― " ―― $r = a$



$$\mathbb{B}_1 = \frac{\mu_0 I_1}{2\pi r_1}\hat{\alpha} = \frac{\mu_0 J_0 \pi r_1^2}{2\pi r_1}\hat{\alpha} \qquad (I_1 = J_0 \pi r_1^2)$$

$$\mathbb{B}_2 = \frac{\mu_0 I_2}{2\pi r_2}\hat{\theta} = \frac{\mu_0 J_0 \pi r_2^2}{2\pi r_2^2} \qquad (I_2 = J_0 \pi r_2^2)$$

$$\hat{\alpha} = \cos\alpha\,\hat{y} - \sin\alpha\,\hat{x}$$

$$\hat{\theta} = \cos\theta\,\hat{y} + \sin\theta\,\hat{x}$$

$$\mathbb{B} = \mathbb{B}_1 + \mathbb{B}_2 = \frac{\mu_0 J_0}{2}\left[r_1\cos\alpha\,\hat{y} - \cancel{r_1\sin\alpha\,\hat{x}} + r_2\cos\theta\,\hat{y} + \cancel{\underbrace{r_2\sin\theta}_{=r_1\sin\alpha}\hat{x}}\right]$$

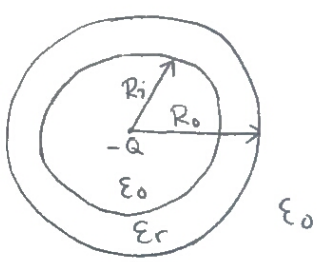
$$r_2\cos\theta = d - r_1\cos\alpha$$

$$\Rightarrow \mathbb{B} = \frac{\mu_0 J_0}{2}\left[r_1\cos\alpha\,\hat{y} + (d - r_1\cos\alpha)\hat{y}\right] = \frac{\mu_0 J_0}{2} d\,\hat{y}$$

Dugga 19/11-11

1.

A charge $-Q$ located in center of a dielectric sphere with permittivity ε_r. Find E, V, D and P as function or radius.



1. Find E and D by Gauss law
2. Find P by $P = D - \varepsilon_0 E = \varepsilon_0(\varepsilon_r - 1)E$
3. Find V by integration of E.

$R > R_o$:

$$E_1 = \frac{-Q}{4\pi\varepsilon_0 R^2} \qquad D_1 = \varepsilon_0 E_1 = \frac{-Q}{4\pi R^2} \qquad V_1 = \frac{-Q}{4\pi\varepsilon_0 R} \qquad P = 0$$

$R_i < R < R_o$:

$$E_2 = \frac{-Q}{4\pi\varepsilon_0\varepsilon_r R^2} \qquad D_2 = \varepsilon_0\varepsilon_r E = \frac{-Q}{4\pi R^2}$$

$$V_2 = -\int_{\infty}^{R_o} E_1\,dR - \int_{R_o}^{R} E_2\,dR = V_1(R=R_o) + \frac{Q}{4\pi\varepsilon_0\varepsilon_r}\int_{R_o}^{R}\frac{1}{R^2}\,dR =$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left[\left(1-\frac{1}{\varepsilon_r}\right)\frac{1}{R_0}+\frac{1}{\varepsilon_r R}\right]$$

$$P_2 = (1 - 1/\varepsilon_r)\frac{-Q}{4\pi R^2}$$

$R < R_i$:

$$E_3 = \frac{-Q}{4\pi\varepsilon_0 R^2} \qquad D_3 = \frac{-Q}{4\pi R^2}$$

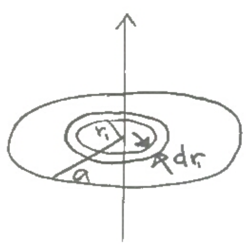
$$V_3 = V_2(R=R_i) - \int_{R_i}^{R} E_3\,dR = \frac{-Q}{4\pi\varepsilon_0}\left[\left(1-\frac{1}{\varepsilon_r}\right)\frac{1}{R_0} - \left(1-\frac{1}{\varepsilon_r}\right)\frac{1}{R_i} + \frac{1}{R}\right]$$

$$P_3 = 0$$

2.

En tunn cirkulär metallskiva (radien=a) befinner sig i vakuum.

$\rho_s(r) = \frac{Q}{2\pi a\sqrt{a^2-r^2}}$ Beräkna kapacitansen till ∞ hos skivan.



$$\begin{cases} C = \frac{Q}{\Delta V} \\ V(\infty) = 0 \end{cases}$$

$$dq = \rho_s(r)\,2\pi r_1 dr_1 = \frac{Q}{2\pi a\sqrt{a^2-r^2}}\,2\pi r_1 dr_1$$

$\mathbb{R}_1 = r_1\hat{r}$ source point $\quad \mathbb{R}_2 = 0$ field point $\Longrightarrow \begin{cases} \mathbb{R}_{12} = \mathbb{R}_2 - \mathbb{R}_1 = -r_1\hat{r} \\ |\mathbb{R}_{12}| = r \end{cases}$

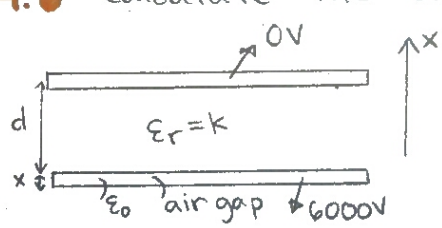
$$V(\mathbb{R}_2) = \int_S \frac{dq}{4\pi\varepsilon_0 R_{12}} = \int_{r_1=0}^{a} \frac{Q}{4\pi\varepsilon_0 a}\left[\frac{dr_1}{\sqrt{a^2-r_1^2}}\right] = \frac{Q}{4\pi\varepsilon_0 a}\left[\arcsin\left(\frac{r_1}{a}\right)\right]_0^a =$$

$$= \frac{Q}{4\pi\varepsilon_0 a}\left(\underbrace{\arcsin(1)}_{=\pi/2} - \underbrace{\arcsin(0)}_{=0}\right) = \frac{Q}{8\varepsilon_0 a}$$

$$\Longrightarrow C = \frac{Q}{\Delta V} = 8\varepsilon_0 a$$

Exempelsamling

4.6 Calculate the force per unit area of the workpiece.



$F_v = \nabla W_e$ fixed voltage system

$$= \frac{\partial}{\partial x} W_e = \frac{\partial}{\partial x}\left(\frac{1}{2}CV^2\right) = \frac{1}{2}V^2\frac{\partial}{\partial x}(C)$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\begin{cases} C_1 = \varepsilon_0 k \dfrac{S}{d} \\ C_2 = \varepsilon_0 \dfrac{S}{x} \end{cases} \Longrightarrow C = \frac{C_1 C_2}{C_1 + C_2} = \frac{\varepsilon_0 k S}{xk + d}$$

$$F_V = \frac{1}{2} V^2 \frac{\partial}{\partial x}\left(\frac{\varepsilon_0 k S}{xk+d}\right) = \frac{1}{2} V^2 \varepsilon_0 k S \times \frac{-k}{(\cancel{xk}+d)^2}$$

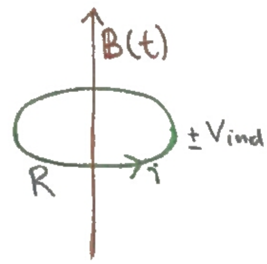
x small

$$\frac{F_V}{S} = \frac{1}{2} V^2 \varepsilon_0 k^2 / d^2 \quad \left[\frac{N}{m^2}\right]$$

Föreläsning 26/11-13

Induktion

Faradays induktionslag: $\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t}$ (postulat)



$$\oint \mathbb{E}\, d\ell = -\int \frac{\partial \mathbb{B}}{\partial t}\, dS$$

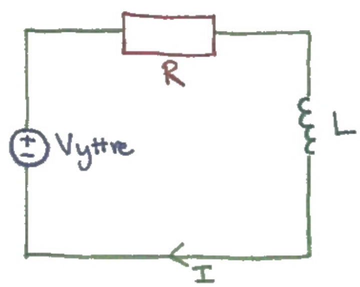
$$V_{ind} = -\frac{\partial \Phi}{\partial t}$$

$$\Phi = \Phi_{totalt} = \Phi_{yttre} + \Phi_{eget}$$

$V_{ind} - RI = 0$

I en krets: $\Phi_{eget} = LI$

$$V_{ind} = \frac{R}{L}\Phi_{eget} = \frac{-\partial \Phi_{totalt}}{\partial t}$$



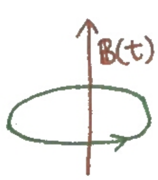
$$\frac{-\partial(\Phi_{yttre} + \Phi_{eget})}{\partial t} = \frac{R}{L}\Phi_{eget} = RI$$

$$\boxed{V_{yttre} = \frac{-\partial \Phi_{yttre}}{\partial t} = RI + \frac{\partial (LI)}{\partial t} = RI + L\frac{\partial I}{\partial t}}$$

Tre fall

1. Fix slinga i tidsvarierande fält
2. Ledare i rörelse i statiskt fält
3. Rörlig ledare i tidsvarierande fält.

1.



$$V_{ind} = \int \nabla \times \mathbb{E}\, dS = \oint \mathbb{E}\cdot d\ell = -\int \frac{\partial \mathbb{B}}{\partial t}\, dS = \begin{Bmatrix}\text{statisk}\\ \text{slinga}\end{Bmatrix} =$$
$$= -\frac{\partial}{\partial t}\int \mathbb{B}\, dS = -\frac{\partial \Phi}{\partial t}$$

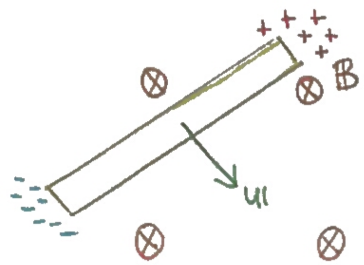
Studera postulatet: $\nabla \times \mathbb{E} = \frac{-\partial \mathbb{B}}{\partial t}$, $\mathbb{B} = \nabla \times \mathbb{A}$, $\nabla \times \mathbb{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbb{A})$

$\nabla \times \mathbb{E} = -\nabla \times \frac{\partial \mathbb{A}}{\partial t}$, $\nabla \times \left(\mathbb{E} + \frac{\partial \mathbb{A}}{\partial t}\right) = 0$ forts. →

Inför potential: $\mathbb{E} + \frac{\partial \mathbb{A}}{\partial t} = -\nabla V$

$\mathbb{E} = \underbrace{-\nabla V}_{\text{laddningar}} \underbrace{- \frac{\partial \mathbb{A}}{\partial t}}_{\text{tidsvarierande strömmar}}$ (annat sätt att skriva det på).

(2.)



Kraft: $\mathbb{F}_m = q(\mathbb{u} \times \mathbb{B})$

Laddningar i vila på ledaren:

$-\nabla V + \frac{\mathbb{F}_m}{q} = 0$

$\Longrightarrow \nabla V = \mathbb{u} \times \mathbb{B}$

(ex 7.3 hemma)

I labbsystemet: $V_2 - V_1 = \int_1^2 \nabla V d\ell = \int_1^2 \mathbb{u} \times \mathbb{B} d\ell$

(3.) Kraft på laddning p.g.a $\mathbb{E}$ & $\mathbb{B}$-fält: $\mathbb{F} = q(\mathbb{E} + \mathbb{u} \times \mathbb{B}) =$

$= \left\{ \begin{matrix} \text{en observatör} \\ \text{som ökar med } q \end{matrix} \right\} = q(\mathbb{E}' + \mathbb{O} \times \mathbb{B}') = q\mathbb{E}'$

$\Longrightarrow \mathbb{E}' = \mathbb{E} + \mathbb{u} \times \mathbb{B}$ $\quad \mathbb{E}'$ i rörligt system

P.s.s $\mathbb{B}' = \mathbb{B} - \frac{1}{c^2}(\mathbb{u} \times \mathbb{E}$

$V_{ind} = \oint_c \mathbb{E}' d\ell = \oint_c \mathbb{E} d\ell + \oint_c \mathbb{u} \times \mathbb{B} d\ell = \int_S \frac{\partial \mathbb{B}}{\partial t} dS + \oint_c \mathbb{u} \times \mathbb{B} d\ell =$

$= V_{ind}^{trans} + V_{ind}^{rörelse} = -\frac{\partial \Phi}{\partial t}$

<u>Maxwells ekvationen kap 7.3</u>

$\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t}$ Faraday

$\nabla \times \mathbb{H} = \mathbb{J}$ Ampere

$\nabla \cdot \mathbb{D} = \rho$ Gauss

$\nabla \cdot \mathbb{B} = 0$

$\nabla \cdot \mathbb{J} + \frac{\partial \rho}{\partial t} = 0$ Kont. ekv.

Är de konsistenta?

$$\underbrace{\nabla\cdot(\nabla\times\mathbb{H})}_{=0} \neq \underbrace{\nabla\cdot\mathbb{J}}_{=-\frac{\partial\rho}{\partial t}}$$

Vi behöver: $\nabla\cdot(\nabla\times\mathbb{H}) = \nabla\cdot\mathbb{J} + \frac{\partial\rho}{\partial t} \equiv 0$

Men $\nabla\cdot\mathbb{D} = \rho$

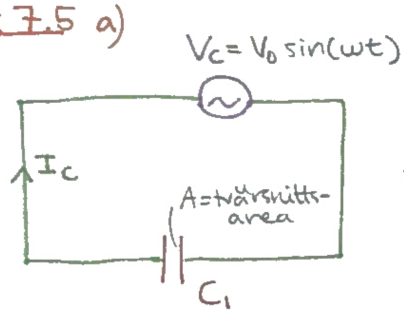
$$\nabla\cdot(\nabla\times\mathbb{H}) = \nabla\cdot\mathbb{J} + \frac{\partial(\nabla\cdot\mathbb{D})}{\partial t} = \nabla\cdot\left(\mathbb{J} + \frac{\partial\mathbb{D}}{\partial t}\right)$$

$$\Longrightarrow \nabla\times\mathbb{H} = \mathbb{J} + \frac{\partial\mathbb{D}}{\partial t}$$ förskjutningsström

(På integralform 7.3.1)

---

ex 7.5 a)



$$I_C = C_1 \frac{\partial V}{\partial t} = \omega C_1 V_0 \cos(\omega t)$$

Fält mellan plattorna:

$$\mathbb{D} = \varepsilon\mathbb{E} = \frac{V_0\sin(\omega t)}{d}\cdot\varepsilon$$

Förskjutningsström: $I_D = \int_A \frac{\partial\mathbb{D}}{\partial t}\,dS = \varepsilon\frac{A}{d}V_0\omega\cos(\omega t) =$

$= \omega C_1 V_0 \cos(\omega t) = I_C$

b) hemma.

Storgruppsövning 26/11-13

Time varying fields and Maxwell's equation kap 7

Fundamental rule for electromagnetic induction:

Faraday's law

$\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t} \xrightarrow[\text{integral}]{\text{surface}} \oint_c \mathbb{E}\, d\ell = -\underbrace{\int_s \frac{\partial \mathbb{B}}{\partial t} dS}_{=\Phi}$

- Stationary circuit in a time-varying $\underbrace{\text{magnetic field}}_{\mathbb{B}(t)}$:

$V = \oint_c \mathbb{E} \cdot d\ell$ induced emf in circuit

$\phi = \int_s \mathbb{B} \cdot dS$ magnetic flux

$V = -\frac{d\Phi}{dt}$ negative rate of increasing magnetic flux

- Moving conductor in a $\underbrace{\text{time-varying magnetic field}}_{\mathbb{B}(t)}$:

$V' = \frac{-d\Phi}{dt}$ emf induced in circuit.

- Moving conductor in a static magnetic field:

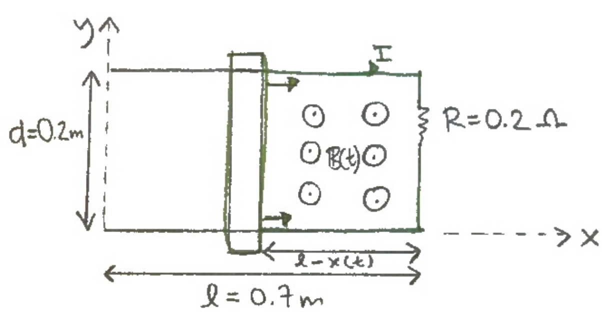
$\frac{\mathbb{E}_m}{q} = u \times \mathbb{B}$ induced electric field

$V_{12} = \int_1^2 (u \times \mathbb{B}) \cdot d\ell$ $\quad V_{12} = \oint_c (u \times \mathbb{B}) \cdot d\ell$ when moving conductor is a part of closed path C.

P 7.7

A conducting bar oscillates over tub parallell-conducting rails in a varying magnetic field.

$\mathbb{B}(t) = \hat{a_z}\, 5 \cos(\omega t)$



Position of bas: $x(t) = \underbrace{0.35}_{\ell/2}(1-\cos(\omega t))$

Find I!

$V = \frac{-d\Phi}{dt}$

$\Phi(t) = \int_s \mathbb{B} \cdot dS = B_z(t) \cdot (\ell - x(t)) \cdot d = \frac{B_0 \ell d}{2} \cos(\omega t)(1+\cos(\omega t))$

$$\phi(t) = \frac{B_0 l d}{2}\left(\cos(\omega t) + \cos^2(\omega t)\right)$$

källa riktning B-fältet

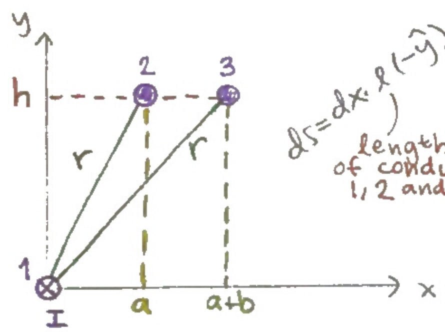
$$V = -\frac{d\phi}{dt} = \frac{B_0 l d}{2}\,\omega \sin(\omega t)\left(1 + 2\cos(\omega t)\right)$$

$V = (-)Ri$ induce current opposes the charge of the magnetic flux (ϕ).

$$I = \frac{-V}{R} = -\frac{B_0 l d}{2R}\,\omega \sin(\omega t)\left(1 + 2\cos(\omega t)\right) = -1{,}75 \cdot 10^{-3}\,\omega \sin(\omega t)\left(1 + 2\cos(\omega t)\right)$$

10.2

Three very long parallell conductor current $I = I_0 \cos(\omega t)$ in conductor 1. Find induced voltage between 2. and 3.



$dS = dx \cdot l(-\hat{y})$ length of conductor 1, 2 and 3.

$$V_{23} = \frac{-d\phi}{dt}$$

$\phi = \int_S \mathbb{B}\, dS$, $\mathbb{B} = -\hat{\phi}\frac{\mu_0 I}{2\pi r}$ magnetic field by current I

$\mathbb{r} = x\hat{x} + h\hat{y}$, $r = \sqrt{x^2 + h^2}$

$$\hat{\phi} = \hat{z} \times \hat{r} = \hat{z} \times \frac{\mathbb{r}}{r} = \hat{z} \times \frac{x\hat{x} + h\hat{y}}{\sqrt{x^2+h^2}} = \frac{x\hat{y} - h\hat{x}}{\sqrt{x^2+h^2}}$$

$$\phi = \int_a^{a+b} -\left(\frac{x\hat{y} - h\hat{x}}{\sqrt{x^2+h^2}}\right)\frac{\mu_0 I}{2\pi\sqrt{x^2+h^2}} \cdot l\, dx(-\hat{y})$$

$$\phi = \frac{\mu_0 I l}{2\pi}\int_a^{a+b}\frac{x}{x^2+h^2}\,dx = \frac{\mu_0 I l}{2\pi}\left[\frac{1}{2}\ln(x^2+h^2)\right] =$$

$$= \frac{\mu_0 I l}{2\pi}\ln\left(\frac{(a+b)^2 + h^2}{a^2+h^2}\right)$$

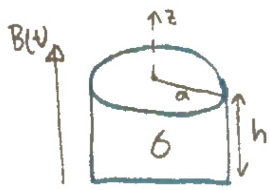
$$V = -d\phi/dt = -\frac{d}{dt}\left[\frac{\mu_0 I_0 l \cos(\omega t)}{4\pi}\ln\left(\frac{(a+b)^2 + h^2}{a^2+h^2}\right)\right] =$$

$$= \frac{\mu_0 I_0 l\, \omega \sin(\omega t)}{4\pi}\left(\ln\left(\frac{(a+b)^2 + h^2}{a^2+h^2}\right)\right)$$

$$\frac{V}{l} = \frac{\mu_0 I_0\, \omega \sin(\omega t)}{4\pi} \cdot \ln\left(\frac{(a+b)^2 + h^2}{a^2+h^2}\right)$$

10.10

A thin conducting disk is in $B(t)$.
$B(t) = B_0\cos(\omega t)$, conductivity σ.
Find the average power dissipation in disk.



$$P = \int_V \sigma E^2 \, dV$$

$$\oint_c E\,dl = -\frac{d\phi}{dt} \xrightarrow[\text{contour with radius } r]{\text{assume a}} 2\pi r E_\phi =$$

$$= \frac{-d}{dt}\underbrace{\int_s B\cdot ds}_{\phi} = -\frac{d}{dt}\left[B_0\cos(\omega t)\pi r^2\right]$$

$$2\cancel{\pi}\cancel{r} E_\phi = B_0\,\omega\sin(\omega t)\cancel{\pi} r^{\cancel{2}}$$

$$\Rightarrow E_\phi = \frac{B_0\omega\sin(\omega t) r}{2}$$

$$P = \int_{r=0}^{a} \sigma\left(\frac{B_0\omega\sin(\omega t) r}{2}\right)^2 \overbrace{2\pi h r dr}^{=dV} = \frac{\sigma B_0^2 \omega^2 \sin^2(\omega t)\pi h}{2}\int_0^a r^3 dr =$$

$$= \frac{\sigma B_0^2\omega^2\sin^2(\omega t)\pi h}{8}\cdot a^4$$

$$\bar{P} = \text{average power dissipation} = \frac{1}{T}\int_0^T P(t)\,dt = \frac{\sigma B_0^2\omega^2\pi h a^4}{8T}\underbrace{\int_0^T \sin^2(\omega t)\,dt}_{T/2} =$$

$$= \frac{\sigma B_0^2 \pi h a^4 \omega}{16}$$

11.2

Use Ohm's law and Maxwell's equations for $\nabla\cdot D$ and $\nabla\times H$ and derive differential equation for $\rho(t)$ and solve this equation.

Maxwell's equations:

$$\nabla\cdot D = \rho \quad \text{free charge density}$$

$$\nabla\times E = \frac{-\partial B}{\partial t}$$

$$\nabla\cdot B = 0$$

$$\nabla\times H = \underbrace{J}_{\text{free charge density}} + \frac{\partial D}{\partial t}$$

Ohm's law: $J = \sigma E$

forts. →

$$\nabla\cdot(\nabla\times\mathbb{H}) = 0 \Longrightarrow \nabla\cdot\left(\mathbb{J} + \frac{\partial\mathbb{D}}{\partial t}\right) = 0$$

$$\Longrightarrow \nabla\cdot\underbrace{\mathbb{J}}_{\sigma\mathbb{E}} + \frac{\partial}{\partial t}\underbrace{(\nabla\cdot\mathbb{D})}_{\rho} = 0$$

$$\Longrightarrow \nabla\cdot(\sigma\mathbb{E}) + \frac{\partial}{\partial t}\rho = 0 \Longrightarrow \frac{\sigma}{\varepsilon}\underbrace{\nabla\cdot(\overbrace{\varepsilon\mathbb{E}}^{\mathbb{D}})}_{\rho} + \frac{\partial\rho}{\partial t} = 0$$

$$\Longrightarrow \frac{\sigma}{\varepsilon}\rho + \frac{\partial\rho}{\partial t} = 0 \Longrightarrow \rho = \rho_0\, e^{-\frac{\sigma}{\varepsilon}t}$$

Föreläsning 27/11-13

Retarderade potentialer 7.4, 7.6

$\mathbb{B} = \nabla \times \mathbb{A}$

$\Rightarrow \nabla \times \mathbb{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbb{A})$ Faradays lag

$\Rightarrow \nabla \times \left(\mathbb{E} + \frac{\partial \mathbb{A}}{\partial t}\right) = 0$

Definieras: $-\nabla V = \mathbb{E} + \frac{\partial \mathbb{A}}{\partial t}$, $\mathbb{E} = -\nabla V - \frac{\partial \mathbb{A}}{\partial t}$

Hur löser vi problemen?

Amperes lag: $\nabla \times \nabla \times \mathbb{A} = \mu_0 \mathbb{J} + \mu\varepsilon \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbb{A}}{\partial t}\right) = \left\{ \begin{array}{l} \nabla \times \nabla \times \mathbb{A} = \\ = \nabla(\nabla \cdot \mathbb{A}) - \nabla^2 \mathbb{A} \end{array} \right\} =$

$= \nabla^2 \mathbb{A} - \mu\varepsilon \frac{\partial^2 \mathbb{A}}{\partial t^2} = -\mu_0 \mathbb{J} + \nabla\left(\nabla \cdot \mathbb{A} + \mu\varepsilon \frac{\partial V}{\partial t}\right)$

Välj $\nabla \cdot \mathbb{A} = -\mu\varepsilon \frac{\partial V}{\partial t} \Longrightarrow \boxed{\nabla^2 \mathbb{A} - \mu\varepsilon \frac{\partial^2 \mathbb{A}}{\partial t^2} = -\mu_0 \mathbb{J}}$ Vågekvationen -dynamikens version av Laplace.

Lösning: $\mathbb{A}(\mathbb{R}, t) = \mathbb{A}(t - R\sqrt{\varepsilon\mu})$

två ggr deriverbar

För V: $\nabla \mathbb{D} = \rho$

$\nabla \cdot \varepsilon\left(-\nabla V - \frac{\partial \mathbb{D}}{\partial t}\right) = \rho$

$\nabla^2 V + \nabla \cdot \frac{\partial \mathbb{A}}{\partial t} = \frac{\rho}{\varepsilon}$

$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho(t)}{\varepsilon}}$

lösningar: $V(\mathbb{R}, t) = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\rho(t - R/u)}{R} dV'$

$A(\mathbb{R}, t) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbb{J}(t - R/u)}{R} dV'$

Inhomogen vågekv. i $\mathbb{E}$ & $\mathbb{H}$ kap 7.6

Faradays lag:

$\nabla \times \nabla \times \mathbb{E} = \nabla \times \left(-\frac{\partial \mathbb{B}}{\partial t}\right) = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbb{H}) = -\mu \frac{\partial}{\partial t}\left(\mathbb{J} + \varepsilon \frac{\partial \mathbb{E}}{\partial t}\right)$

$\nabla(\underbrace{\nabla \cdot \mathbb{E}}_{\rho/\varepsilon}) - \nabla^2 \mathbb{E} = -\mu \frac{\partial \mathbb{J}}{\partial t} - \varepsilon\mu \frac{\partial^2 \mathbb{E}}{\partial t^2}$

$\Longrightarrow \boxed{\nabla^2 \mathbb{E} - \mu\varepsilon \frac{\partial^2 \mathbb{E}}{\partial t^2} = \mu \frac{\partial \mathbb{J}}{\partial t} + \frac{1}{\varepsilon} \nabla \rho}$ Vågekv. för $\mathbb{E}$

$$\boxed{\nabla^2 \mathbb{H} - \mu\varepsilon \frac{\partial^2 \mathbb{H}}{\partial t^2} = -\nabla \times \mathbb{J}}$$ Vågekv. för $\mathbb{H}$

Komplexa fält 7.7

Antag sinusformade fält (i tiden)

$\mathbb{E}(\mathbb{r},t) = \hat{x} E_{ox} \cos[\omega t + \theta_x(r)] + \hat{y} E_{oy} \cos[\omega t + \theta_y(r)] + \hat{z} E_{oz} \cos[\omega t + \theta_z(r)]$

Definiera komplexa fält:

$E(r) = \hat{x} E_{ox}(r) e^{i\theta_x(r)} + \hat{y} E_{oy} e^{i\theta_y} + \hat{z} E_{oz} e^{i\theta_z} =$

$= \hat{x} \bar{E}_{ox} + \hat{y} \bar{E}_{oy} + \hat{z} \bar{E}_{oz}$

Återgå till reellt fält: $\mathbb{E}(\mathbb{r},t) = \mathrm{Re}\{\bar{\mathbb{E}}(r) e^{i\omega t}\}$

Vad händer för fältekv.?

$\nabla \times \left(\mathrm{Re}(\bar{\mathbb{E}} e^{i\omega t})\right) = -\frac{\partial}{\partial t}\left(\mathrm{Re}(\bar{\mathbb{B}} e^{i\omega t})\right)$

$\mathrm{Re}(\nabla \times \bar{\mathbb{E}} e^{i\omega t}) = \mathrm{Re}(-\partial/\partial t\, \bar{\mathbb{B}} e^{i\omega t})$

$\mathrm{Re}(e^{i\omega t} \nabla \times \bar{\mathbb{E}}) = \mathrm{Re}(\mathbb{B}[-i\omega \cdot e^{i\omega t}])$

Måste gälla för alla t:

$\nabla \times \bar{\mathbb{E}} = -i\omega \bar{\mathbb{B}}$

Maxwells ekvationer:

$\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t} \longrightarrow \nabla \times \bar{\mathbb{E}} = -i\omega \mathbb{B}$

$\nabla \cdot \mathbb{B} = 0 \longrightarrow \nabla \cdot \bar{\mathbb{B}} = 0$

$\nabla \times \mathbb{H} = \mathbb{J} + \frac{\partial \mathbb{D}}{\partial t} \longrightarrow \nabla \times \bar{\mathbb{H}} = \bar{\mathbb{J}} + i\omega \bar{\mathbb{D}}$

$\nabla \cdot \mathbb{D} = \rho \longrightarrow \nabla \cdot \bar{\mathbb{D}} = \bar{\rho}$

För vågekv. fås:

$\nabla^2 \mathbb{E} - \sigma\mu \frac{\partial \mathbb{E}}{\partial t} - \varepsilon\mu \frac{\partial^2 \mathbb{E}}{\partial t^2} = 0$

$\nabla^2 \bar{\mathbb{E}} - i\omega\sigma\mu \bar{\mathbb{E}} - (i\omega)^2 \varepsilon\mu \bar{\mathbb{E}} = 0$

brukar skrivas som:

$\nabla^2 \bar{\mathbb{E}} - \gamma^2 \bar{\mathbb{E}} = 0$, $\gamma = \alpha + i\beta = \sqrt{i\omega\mu(i\omega\varepsilon + \sigma)}$ utbredningskonstant

Plan våg 8.1, 8.2

Utbredningsriktning (t.ex $\hat{z}$-led)

Fältstyrkan vid fix tidpunkt är konstant till storlek och riktning i ett oändligt plan vinkelrätt mot utbredningsriktningen

Ansätt plan våg:

$$\bar{E}(z) = \hat{x}\bar{E}_x(z) + \hat{y}\bar{E}_y(z) + \hat{z}\bar{E}_z(z)$$

$$\bar{H}(z) = \hat{x}\bar{H}_x(z) + \hat{y}\bar{H}_y(z) + \hat{z}\bar{H}_z(z)$$

Koll: $\rho_{fri} = 0 \Rightarrow \nabla \cdot \bar{D} = 0 \quad \nabla \cdot \bar{B} = 0$

Utför divergens: $\frac{\partial \bar{E}_z}{\partial z} = 0 \quad \Big| \quad \frac{\partial \bar{H}_z}{\partial z} = 0$

$\Rightarrow \bar{E}_z =$ konstant

$\Rightarrow \bar{H}_z =$ konstant

För vågor är E_z och H_z ej av intresse

ex. (på notation)

$E(z) = E_0 e^{-j\mu z}\hat{x}$

Polarisation 8.2.3

Allmän plan våg: $\bar{E} = \hat{x}E_{x0}\cos(\omega t - \beta z) + \hat{y}E_{y0}\cos(\omega t - \beta z + \varphi)$

a) linjärt polariserad om $\varphi = \pm k\pi$

b) cirkulärpolariserad om $E_{x0} = E_{y0}$ och $\varphi = \pm(k + \frac{1}{2})\pi$

c) annars elliptisk

Storgruppsövning 27/11-13

Source free wave equations

$$\left(\begin{array}{ll} \nabla \times \mathbb{E} = -\mu \frac{\partial \mathbb{H}}{\partial t} & \nabla \cdot \mathbb{E} = 0 \\ \nabla \times \mathbb{H} = \varepsilon \frac{\partial \mathbb{E}}{\partial t} & \nabla \cdot \mathbb{H} = 0 \end{array}\right) \text{ Maxwell's equations } \Longrightarrow \begin{cases} \nabla^2 \mathbb{E} - \frac{1}{u^2}\frac{\partial^2 \mathbb{E}}{\partial t^2} = 0 \\ \nabla^2 \mathbb{H} - \frac{1}{u^2}\frac{\partial^2 \mathbb{H}}{\partial t^2} = 0 \end{cases} \text{ homogenous wave equations}$$

u — speed of wave

Phasors:

time harmonic $\mathbb{E}$-field $\mathbb{E}(x,y,z,t) = \mathrm{Re}\{\overline{\mathbb{E}} e^{i\omega t}\}$ refers to cos

$\overline{\mathbb{E}}$ — vector phasor depends on space coordinates

$$\left(\begin{array}{ll} \nabla \times \overline{\mathbb{E}} = -i\omega\mu \overline{\mathbb{H}} & \nabla \cdot \overline{\mathbb{E}} = 0 \\ \nabla \times \overline{\mathbb{H}} = \underbrace{i\omega}_{\partial/\partial t} \varepsilon \overline{\mathbb{E}} & \nabla \cdot \overline{\mathbb{H}} = 0 \end{array}\right)$$

$$\left\{\begin{array}{l} \nabla^2 \overline{\mathbb{E}} + k^2 \overline{\mathbb{E}} = 0 \\ \nabla^2 \overline{\mathbb{H}} + k^2 \overline{\mathbb{H}} = 0 \end{array}\right\} \text{ homogenous vector Helmholtz equations}$$

$$k = \omega\sqrt{\mu\varepsilon} = \omega/u$$

k — wave number

12.3

an electromagnetic wave in vacuum, ω angular freq.

$\mathbb{E} = \hat{y} E_0 e^{-\alpha z} e^{-i\beta x} = \hat{y}\overline{E}_y$

a) Find $\mathbb{H}$ and $\mathbb{E}$ in real format

$E_y(t) = \mathrm{Re}\{\overline{E}_y e^{i\omega t}\} = E_0 e^{-\alpha z}\cos(\omega t - \beta x)$

$$\overline{\mathbb{H}} = \frac{-1}{i\omega\mu_0}\nabla \times \overline{\mathbb{E}} = \frac{-1}{i\omega\mu_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \overline{E}_y & 0 \end{vmatrix} = \frac{-1}{i\omega\mu_0}\left(-\hat{x}\frac{\partial \overline{E}_y}{\partial z} + \hat{z}\frac{\partial \overline{E}_y}{\partial x}\right)$$

$$\overline{\mathbb{H}} = \frac{-1}{i\omega\mu_0}\left(-\hat{x}(-\alpha E_0 e^{-\alpha z} e^{-i\beta x}) + \hat{z}(-i\beta E_0 e^{-\alpha z} e^{-i\beta x})\right) =$$

$$= \frac{i\overline{E}_y}{\omega\mu_0}\left[\hat{x}\alpha - \hat{z} i\beta\right] \Longrightarrow H_x(t) = \mathrm{Real}\{\overline{H}_x e^{i\omega t}\} = \frac{E_0\alpha}{\omega\mu_0} e^{-\alpha z}\cos(\omega t - \beta x + \tfrac{\pi}{2})$$

$$H_z(t) = \mathrm{Real}\{\overline{H}_z e^{i\omega t}\} = \frac{E_0\beta}{\omega\mu_0} e^{-\alpha z}\cos(\omega t - \beta x)$$

b) What is the relation between α, β and ω to satisfy wave equation? In vacuum we have: Maxwell's eq.

forts. →

$$\nabla\times(\nabla\times\bar{E}) = -i\omega\mu_0(\nabla\times\bar{H}) = -i\omega\mu_0(i\omega\varepsilon_0\bar{E}) = \omega^2\underbrace{\mu_0\varepsilon_0}_{1/c^2}\bar{E}$$

$$\nabla(\underbrace{\nabla\cdot\bar{E}}_{=0}) - \nabla^2\bar{E} = \omega^2\mu_0\varepsilon_0\bar{E}$$

$$\Rightarrow \nabla^2\mathbb{E} + \frac{\omega^2}{c^2}\mathbb{E} = 0 \quad , \quad c = 1/\sqrt{\mu_0\varepsilon_0}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\mathbb{E} + \frac{\omega^2}{c^2}\mathbb{E} = 0 \quad , \quad \mathbb{E} = \mathbb{E}_0 e^{-\alpha z} e^{-i\beta x}$$

$$\Longrightarrow (-i\beta)^2 + 0 + (-\alpha)^2 + \frac{\omega^2}{c^2} = 0 \Rightarrow \beta^2 - \alpha^2 = \frac{\omega^2}{c^2}$$

Another way to solve it:

$$\mathbb{E} = \hat{y}E_0 e^{-\alpha z}e^{-i\beta x} = \hat{y}\mathbb{E}_0 e^{-i\mathbb{k}\mathbb{R}}$$

$$\mathbb{R} = \hat{x}x + \hat{y}y + \hat{z}z$$ $\mathbb{R}$, radius vector from origin

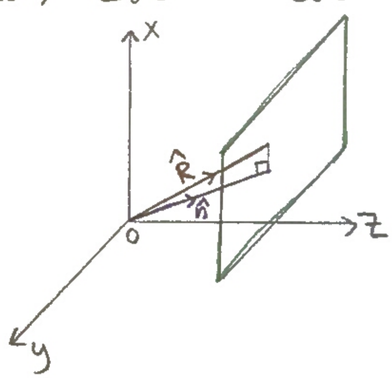
$$-i(\mathbb{k}\cdot\mathbb{R}) = -\alpha z - i\beta x$$

$$\Longrightarrow \mathbb{k} = \hat{x}\beta - \hat{z}(i\alpha) = k_x\hat{x} + k_z\hat{z}$$

$$k^2 = k_x^2 + k_z^2 = \beta^2 - \alpha^2 = \frac{\omega^2}{c^2}$$

Uniform plane wave in lossless media

$$\bar{E}(\mathbb{R}) = \bar{E}_0 e^{-i\mathbb{k}\cdot\mathbb{R}} = E_0 e^{-ik\hat{n}\mathbb{R}}$$



$\mathbb{R}$: radius vector from origin
$\hat{n}$: direction of propagation

$k = \omega\sqrt{\mu\varepsilon}$, wave number
$\mathbb{k} = k\hat{n} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$
$|k|^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon$

$$\bar{H}(R) = \frac{-1}{i\omega\mu}\nabla\times\bar{E}(R) = \frac{1}{\eta}\hat{n}\times\bar{E}(R) \quad , \quad \overset{\text{wave impedance}}{\eta} = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\bar{E}(R) = \frac{1}{i\omega\varepsilon}\nabla\times\bar{H}(R) = -\eta\,\hat{n}\times\bar{H}(R) \quad , \quad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 120\pi$$

11.8

A plane sinusoidal wave in vacuum.

$$\begin{cases} H_x = A\cos\omega\left(t - \frac{1}{c}(y\sin\alpha + z\cos\alpha)\right) = \mathrm{Re}\{\bar{H}e^{i\omega t}\} \\ H_y = H_z = 0 \end{cases}$$

Find $\mathbb{E}$!

$$\bar{\mathbb{H}} = \hat{x}Ae^{-i\frac{\omega}{c}(y\sin\alpha + z\cos\alpha)} = \hat{x}Ae^{-ik\hat{n}\mathbb{R}} = \hat{x}H_x$$

$$\mathbb{R} = \hat{x}x + \hat{y}y + \hat{z}z$$

$$\Rightarrow \begin{cases} k = \frac{\omega}{c} \\ \hat{n} = \hat{y}\sin\alpha + \hat{z}\cos\alpha \end{cases}$$ unit vector in direction of propagation

$$\bar{\mathbb{E}}(R) = -\eta\hat{n}\times\bar{\mathbb{H}}(R) = \left\{\eta = \frac{\omega\mu_0}{k} = \frac{k}{\omega\varepsilon_0}\right\} = \frac{-k}{\omega\varepsilon_0}(\hat{y}\sin\alpha + \hat{z}\cos\alpha)\times\hat{x}Ae^{-ik\hat{n}\cdot R} =$$

$$= \frac{-k}{\omega\varepsilon_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \sin\alpha & \cos\alpha \\ \bar{H}_x & 0 & 0 \end{vmatrix} = \frac{-k}{\omega\varepsilon_0}\left(\hat{y}\cos\alpha - \hat{z}\sin\alpha\right)\bar{H}_x =$$

$$= -\eta_0(\hat{y}\cos\alpha - \hat{z}\sin\alpha)Ae^{-i\frac{\omega}{c}(y\sin\alpha + z\cos\alpha)}$$

$$\begin{cases} E_y(t) = -\eta_0\cos\alpha A\cos\left(\omega t - \frac{\omega}{c}(y\sin\alpha + z\cos\alpha)\right) = \mathrm{Re}\{\bar{E}e^{i\omega t}\} \\ E_z(t) = \eta_0\sin\alpha A\cos\left(\omega t - \frac{\omega}{c}(y\sin\alpha + z\cos\alpha)\right) \end{cases}$$

Another way to solve the problem:

$$\mathbb{E} = \frac{1}{i\omega\varepsilon_0}\nabla\times\mathbb{H} = \frac{1}{i\omega\varepsilon_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \bar{H}_x & 0 & 0 \end{vmatrix} \quad \ldots$$

Plane waves in lossy media

In a source free: $\nabla^2\mathbb{E} + k_c^2 + \mathbb{E} = 0$

complex number

$k_c = \omega\sqrt{\mu\varepsilon_c}$ complex wave number

$$\gamma = ik_c = i\omega\sqrt{\mu\varepsilon_c} = \alpha + i\beta$$

propagation constant (γ), attenuation constant (α), phase constant (β)

$$\nabla^2 \mathbb{E} - \gamma^2 \mathbb{E} = 0$$ wave eq. in lossy media

Solution $\Longrightarrow$ $\bar{\mathbb{E}} = \hat{x}\bar{E}_x = \hat{x}E_0 e^{-\gamma z} = \hat{x}E_0 e^{-\alpha z} e^{-i\beta z}$

Good conductors: $\sigma/\omega\varepsilon \gg 1$, $\gamma = \alpha + i\beta = (1+i)\sqrt{\frac{\omega\mu\sigma}{2}}$

12.7

Calculate α, β, $Z^{=\eta}$ for a metal with permeability μ_r and conductivity σ, $\sigma \gg \omega\varepsilon_0\mu_r$

$$\nabla \times \bar{\mathbb{H}} = \bar{\mathbb{J}} + i\omega\bar{\mathbb{D}} = \sigma\bar{\mathbb{E}} + i\omega\varepsilon\bar{\mathbb{E}} = i\omega\underbrace{\left(\varepsilon + \frac{\sigma}{i\omega}\right)}_{\varepsilon_c = \varepsilon - i\frac{\sigma}{\omega}}\bar{\mathbb{E}} = i\omega\varepsilon_c\bar{\mathbb{E}}$$

$$\nabla^2 \mathbb{E} - \gamma^2 \mathbb{E} = 0 \quad , \gamma = ik_c = i\omega\sqrt{\mu\varepsilon_c} = i\omega\sqrt{\mu(\underset{\text{neglect}}{\cancel{\varepsilon}} - i\frac{\sigma}{\omega})} = i\omega\sqrt{\mu(-i\frac{\sigma}{\omega})} =$$

$$= \sqrt{i\sigma\omega\mu} = \frac{1+i}{\sqrt{2}}\sqrt{\omega\mu\sigma} = \alpha + \beta i$$

$$\Longrightarrow \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$Z_c = \eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\cancel{\varepsilon} - i\frac{\sigma}{\omega}}} = \sqrt{\frac{\mu}{-i\frac{\sigma}{\omega}}} = \sqrt{\frac{i\omega\mu}{\sigma}} \quad \left(\cdot\sqrt{\frac{\mu_0}{\varepsilon_0}}\cdot\sqrt{\frac{\varepsilon_0}{\mu_0}}\right)$$

$$\eta_c = \frac{1+i}{\sqrt{2}}\underbrace{\sqrt{\frac{\mu_0}{\varepsilon_0}}}_{\eta_0}\sqrt{\frac{\omega\mu\varepsilon_0}{\mu_0\sigma}} = (1+i)\eta_0\sqrt{\frac{\omega\mu_r\varepsilon_0}{2\sigma}} \quad \measuredangle Z_c = 45^\circ$$

- the magnetic field lags behind the E-field by 45°.

$$|\eta_c| = \frac{|\mathbb{E}|}{|\mathbb{H}|}$$

Föreläsning 29/11-13

Studera en plan våg som propagerar i z-led

$\mathbb{E} = \hat{x}\,\bar{E}_x(z)$

Vågekv.: $\frac{\partial^2 \bar{E}_x(z)}{\partial z^2} - \gamma^2 E_x(z) = 0$

lösning: $\bar{E}_x(z) = \bar{E}_x^-(0)e^{-\gamma z} + \bar{E}_x^+(0)e^{\gamma z}$

Där $\bar{E}_x^+ = E_x^+(0)e^{i\theta+}$

$\bar{E}_x^- = E_x^-(0)e^{i\theta-}$

På reell form: $E_x(z,t) = \mathrm{Re}\{\bar{E}_x(z)e^{i\omega t}\} = \mathrm{Re}\{(E_x^-(0)e^{i\theta^-}e^{-(\alpha+i\beta)z} +$

$+ E_x^+(0)e^{i\theta^+}e^{(\alpha+i\beta)z})\,e^{i\omega t}\} = E_x^-(0)\,e^{-\alpha z}\cos(\omega t - \beta z + \theta^-) +$

$+ E_x^+(0)\,e^{\alpha z}\cos(\omega t + \beta z + \theta^+)$

Två vågor som utbreder sig i motsatt riktning

α - dämpning
β - faskonstant

<u>Fashastighet</u>
Följ t.ex en vågtopp
$\omega t - \beta z + \theta^+ = $ konstant
Derivera m.a.p t: $\omega - \beta\frac{dz}{dt} = 0$

$\Rightarrow v_{fas} = \frac{dz}{dt} = \frac{\omega}{\beta}$

<u>Vågimpedans kap 8.2.2</u>
Relation mellan $\mathbb{B}$ och $\mathbb{E}$ (η boken, z här)

Ur postulatet: $\nabla \times \bar{\mathbb{E}} = -i\omega\bar{\mathbb{B}}$ Fås för $\bar{\mathbb{E}} = \hat{x}\,\bar{E}_x(z)$

$\bar{\mathbb{H}} = \frac{-1}{i\omega\mu}\left[\hat{y}\frac{\partial \bar{E}_x}{\partial z}\right] = \frac{-\hat{y}}{i\omega\mu}\left[-\gamma\bar{E}_x^-(z) + \gamma\bar{E}_x^+(z)\right] = \frac{\hat{y}\gamma}{i\omega\mu}\left[\bar{E}_x^-(z) - \bar{E}_x^+(z)\right] =$

$= \hat{y}\left[\bar{H}_y^-(z) + \bar{H}_y^+(z)\right]$

Identifiera: $\begin{cases} \bar{H}_y^+(z) = \frac{-\gamma}{i\omega\mu}E_x^+ = -\frac{1}{z}\bar{E}_x^+(z) \\ \bar{H}_y^-(z) = \frac{\gamma}{i\omega\mu}\bar{E}_x^- = \frac{1}{z}\bar{E}_x^-(z) \end{cases}$ $\quad z = \frac{i\omega\mu}{\gamma} = \sqrt{\frac{i\omega\mu}{i\omega\varepsilon+\sigma}} = \frac{\gamma}{i\omega\varepsilon+\sigma}$

$\gamma = \sqrt{i\omega\mu(i\omega\varepsilon+\sigma)} =$
$= \alpha + i\beta$

För plan våg: $\mathbb{E}(\mathbf{r}) = \mathbb{E}(0)\, e^{-\gamma \mathbf{k}\cdot\mathbf{r}}$, $\hat{\mathbf{k}}\,\mathbb{E} = 0$

propagations-riktning

Sätt in i $\nabla \times \mathbb{E} = -i\omega\mathbb{B}$

$\Longrightarrow \mathbb{H}(\mathbf{r}) = \frac{1}{z}\,\hat{\mathbf{k}} \times \mathbb{E}(\mathbf{r})$

$\Longleftrightarrow \mathbb{E} = z\,\mathbb{H}(\mathbf{r}) \times \hat{\mathbf{k}}$

Beräkning av α och β kap 8.3

Utgå från $\begin{cases} \gamma^2 = i\omega\mu\sigma - \omega^2\varepsilon\mu \\ \gamma = \alpha + i\beta \quad , \alpha, \beta \geq 0 \end{cases}$

$$\Longrightarrow \alpha = \omega\sqrt{\frac{\varepsilon\mu}{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1} \quad , \quad \beta = \omega\sqrt{\frac{\varepsilon\mu}{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1}$$

För en god ledare: $\frac{\sigma}{\omega\varepsilon} >> 1$, $\alpha \approx \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}}$

För ett dielektrisk material: (t.ex vatten)

$\frac{\sigma}{\omega\varepsilon} << 1$, $\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$, $\beta = \omega\sqrt{\varepsilon\mu}$

Skineffekt kap 8.3

E-fält propagerar in i ett ledande halvplan

$\mathbb{E}(z) = \hat{x}\, E_x^+(0)\, e^{-\gamma z} = \hat{x}\, E_x^+(0)\, e^{i\theta^+} e^{-\alpha z} e^{-i\beta z}$

reell form $\Longrightarrow$ $E(z,t) = \hat{x}\, E_x^+(0)\, \underbrace{e^{-\alpha z}}_{\text{dämpning}} \cos(\omega t - \beta z + \theta^+)$

Fältet dämpas med faktorn $e^{-\alpha z}$

$\delta = 1/\alpha$

inträngningsdjup

På djupet $z = \delta$ har fältstyrkan dämpats till $e^{-1} \approx 37\%$

För en metall $\frac{\sigma}{\omega\varepsilon} >> 1$, $\delta \approx \sqrt{\frac{2}{\omega\mu\sigma}}$

Föreläsning 3/12-13

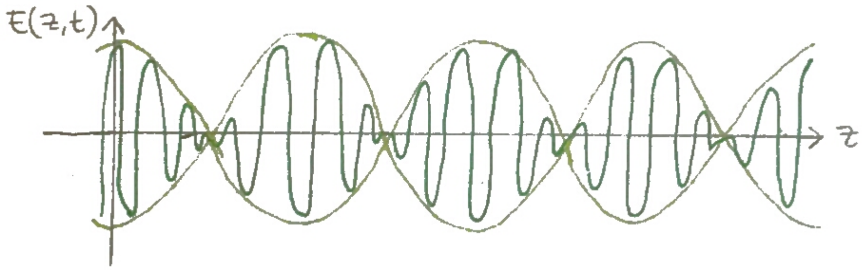
Grupphastighet 8.4

Fashastighet: $V_{fas} = \frac{\omega}{\beta}$

Våglängd: $\lambda = \frac{2\pi}{\beta}$

Betrakta två vågor med olika frekvens:
$\omega_0 + \Delta\omega$ och $\omega_0 - \Delta\omega$
$\beta_0 + \Delta\beta$ och $\beta_0 - \Delta\beta_0$

$$E(z,t) = E_0 \cos[(\omega_0+\Delta\omega)t - (\beta_0+\Delta\beta)z] + E_0\cos[(\omega_0-\Delta\omega)t - (\beta_0-\Delta\beta)z]$$
$$= \dots = 2E_0\cos(\Delta\omega t - \Delta\beta z)\cos(\omega_0 t - \beta_0 z)$$



Grupphastighet: $\Delta\omega t - \Delta\beta z = \text{konstant}$, $V_g = \frac{\partial z}{\partial t} = \frac{\Delta\omega}{\Delta\beta}$

Låt $D \to 0 \Rightarrow V_g = 1 / \frac{\partial\beta}{\partial\omega}$

Om $\beta \propto \omega \Rightarrow V_{grupp} = V_{fas}$

Poyntingvektorn 8.5

$$\nabla\cdot(\mathbb{E}\times\mathbb{H}) = \mathbb{H}(\nabla\times\mathbb{E}) - \mathbb{E}(\nabla\times\mathbb{H}) = \mathbb{H}\left(-\frac{\partial\mathbb{B}}{\partial t}\right) - \mathbb{E}\left(\mathbb{J} + \frac{\partial\mathbb{D}}{\partial t}\right) =$$
$$= \frac{1}{2}\frac{\partial}{\partial t}(\mathbb{H}\cdot\mathbb{B}) - \frac{1}{2}\frac{\partial}{\partial t}(\mathbb{E}\cdot\mathbb{D}) - \frac{\mathbb{J}\cdot\mathbb{J}}{\sigma}$$

Integrera över volym:
$$\underbrace{\int_{V'}\nabla\cdot(\mathbb{E}\times\mathbb{H})\,dV'}_{\text{effekt ut}} + \underbrace{\frac{\partial}{\partial t}\int_{V'} W_m + W_e \, dV'}_{\text{tidsändring av fältenergi}} + \underbrace{\int_{V'}\frac{\mathbb{J}\cdot\mathbb{J}}{\sigma}\,dV'}_{\text{Joules lag, ohmska förluster}} = 0$$

$$\int_{V'}\nabla\cdot(\mathbb{E}\times\mathbb{H})\,dV' = \int_S \mathbb{E}\times\mathbb{H}\, d\mathbb{S}$$

(se ex. 8.7 hemma)

$\mathbb{S} = \mathbb{E}\times\mathbb{H}$ Poyntingvektorn $[W/m^2]$

Komplexa poyntingvektorn

$$\left.\begin{aligned}\mathbb{E}(\mathbf{r}) &= E_{re}(\mathbf{r}) + iE_{im}(\mathbf{r})\\ \mathbb{H}(\mathbf{r}) &= H_{re}(\mathbf{r}) + iH_{im}(\mathbf{r})\end{aligned}\right\}\ \text{Komplexa fält}$$

$$\left.\begin{aligned}\mathbb{E}(\mathbf{r},t) &= E_{re}(\mathbf{r})\cos\omega t - E_{im}(\mathbf{r})\sin\omega t\\ \mathbb{H}(\mathbf{r},t) &= H_{re}(\mathbf{r})\cos\omega t - H_{im}(\mathbf{r})\sin\omega t\end{aligned}\right\}\ \text{Reell form}$$

$$\mathbb{S} = \mathbb{E}\times\mathbb{H} = (E_{re}\cos\omega t - E_{im}\sin\omega t)\times(H_{re}\cos\omega t - H_{im}\sin\omega t) =$$

$$= (E_{re}\times H_{re})\cos^2\omega t + (E_{im}\times H_{im})\sin^2\omega t -$$

$$- [(E_{im}\times H_{re}) + (E_{re}\times H_{im})]\underbrace{\sin\omega t\cos\omega t}_{\text{tidsm.}=0}$$

Tidsmedelvärde:

$$S_{av} = \frac{1}{T}\int_0^T S(r,t)\,dt = \boxed{\frac{1}{2}\left[E_{re}\times H_{re} + E_{im}\times H_{im}\right]}\ \text{Tidsmv. av poyntingvektorn}$$

T: helt antal perioder

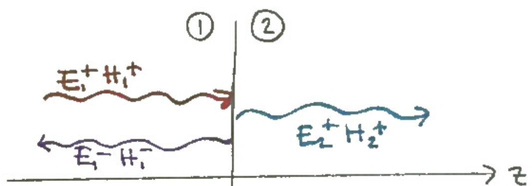
Komplexa fält:

$$\frac{1}{2}\mathrm{Re}\{\mathbb{E}\times\mathbb{H}^*\} = \frac{1}{2}\mathrm{Re}\{(\mathbb{E}_{re}+i\mathbb{E}_{im})\times(\mathbb{H}_{re}-i\mathbb{H}_{im})\} =$$

$$= \frac{1}{2}\{(\mathbb{E}_{re}\times\mathbb{H}_{re}) + (\mathbb{E}_{im}\times\mathbb{H}_{im}) + i(\mathbb{E}_{im}\times\mathbb{H}_{re} - \mathbb{E}_{re}\times\mathbb{H}_{im})\} =$$

$$= \boxed{\frac{1}{2}\left(\mathbb{E}_{re}\times\mathbb{H}_{re} + \mathbb{E}_{im}\times\mathbb{H}_{im}\right)}\ \text{Tidsmv. av komplexa poyntingvektorn.}$$

Reflektion och transmission 8.8, 8.6



$\gamma_1 = \sqrt{i\omega\mu_1(i\omega\varepsilon_1+\sigma_1)}$ | $\gamma_2 = \sqrt{i\omega\mu_2(i\omega\varepsilon_2+\sigma_2)}$

$Z_1 = \sqrt{\dfrac{i\omega\mu_1}{i\omega\varepsilon_1+\sigma_1}}$ | $Z_2 = \sqrt{\dfrac{i\omega\mu_2}{i\omega\varepsilon_2+\sigma_2}}$

forts →

Antag plana vågor som propagerar i z-led.
E-fältet polariserat i $\hat{x}$-led
H ——— " ——— $\hat{y}$-led

$$\bar{E}_1^+ = \hat{x}\,\bar{E}_{10}^+ e^{-\gamma_1 z} \qquad \bar{H}_1^+ = \hat{y}\,(\bar{E}_{10}^+/Z_1)\,e^{-\gamma_1 z}$$

$$\bar{E}_1^- = \hat{x}\,\bar{E}_{10}^- e^{+\gamma_1 z} \qquad \bar{H}_1^- = -\hat{y}\,(\bar{E}_{10}^-/Z_1)\,e^{\gamma_1 z}$$

$$\bar{E}_2^+ = \hat{x}\,\bar{E}_{20}^+ e^{-\gamma_2 z} \qquad \bar{H}_2^+ = \hat{y}\,(\bar{E}_{20}^+/Z_2)\,e^{-\gamma_2 z}$$

Randvillkor ger: $E_{1tang} = E_{2tang}$

$H_{1tang} = H_{2tang}$ (om inga fria strömmar)

Om gränsytan vid $z = 0$

$$\Longrightarrow \bar{E}_{10}^+ + \bar{E}_{10}^- = \bar{E}_{20}^+$$

$$\frac{\bar{E}_{10}^+}{Z_1} - \frac{\bar{E}_{10}^-}{Z_1} = \frac{\bar{E}_{20}^+}{Z_2}$$

Eliminera:

$$\bar{E}_{10}^- = \frac{Z_2 - Z_1}{Z_2 + Z_1}\,\bar{E}_{10}^+ \qquad \Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$\bar{E}_{20}^+ = \frac{2Z_2}{Z_2 + Z_1}\,\bar{E}_{10}^+ \qquad \tau = \frac{2Z_2}{Z_2 + Z_1}$$

Kan visa att $1 + \Gamma = \tau$

För effekt $R = \frac{|S_r|}{|S_i|} = \frac{|\bar{E}_r|^2}{|\bar{E}_i|^2} = |\Gamma|^2$

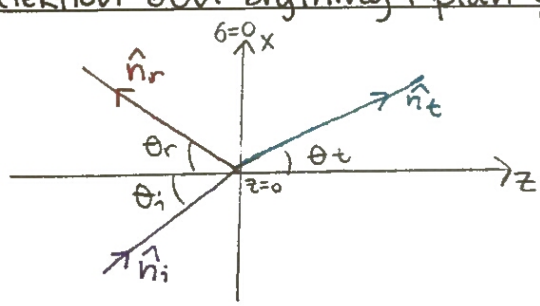
reflekterande effekt

$T = \frac{|S_t|}{|S_i|} = \frac{|S_i| - |S_r|}{|S_i|} = 1 - \frac{|S_r|}{|S_i|} = 1 - R$

transmitterande effekt

$\Longrightarrow R + T = 1$ energiprincipen gäller än

Reflektion och brytning i plan gränsyta 8.10



Ansätt $\bar{E}_i(R) = \bar{E}_{i0}\,e^{-i\beta_1 \hat{n}_i R}$

$\bar{E}_r(R) = \bar{E}_{r0}\,e^{-i\beta_1 \hat{n}_r R}$

$\bar{E}_t(R) = \bar{E}_{t0}\,e^{-i\beta_2 \hat{n}_t R}$

forts. →

$\hat{n}_i = (\sin\theta_i, 0, \cos\theta_i) \Longrightarrow \hat{n}_i \mathbb{R} = x\sin\theta_i + z\cos\theta_i$

$\hat{n}_t = (\sin\theta_t, 0, \cos\theta_t) \Longrightarrow \hat{n}_t \mathbb{R} = x\sin\theta_t + z\cos\theta_t$

$\hat{n}_r = (\sin\theta_r, 0, -\cos\theta_r) \Longrightarrow \hat{n}_r \mathbb{R} = x\sin\theta_r - z\cos\theta_r$

Vid $z=0$ gäller: $(\bar{E}_i + \bar{E}_r)_{tang} = (\bar{E}_t)_{tang}$

$(\bar{H}_i + \bar{H}_r)_{tang} = (\bar{H}_t)_{tang}$

$$\bar{E}_{i0tang}\, e^{-i\beta_1 x\sin\theta_i} + \bar{E}_{r0tang}\, e^{-i\beta_1 x\sin\theta_r} = \bar{E}_{t0tang}\, e^{-i\beta_2 x\sin\theta_t}$$

$(\beta_1 = \omega/c_1)$

Uppfyllt om: $\frac{\omega}{c_1} x\sin\theta_i = \frac{\omega}{c_1} x\sin\theta_r = \frac{\omega}{c_2} x\sin\theta_t$

$$\boxed{\begin{aligned}&\text{Gäller om } \theta_i = \theta_r \\ &c_2\sin\theta_i = c_1\sin\theta_t\end{aligned}}$$

Snells lag

Storgruppsövning 3/12−13

The relationship between the induced emf and the rate of change of flux linkage is known as Faraday's law

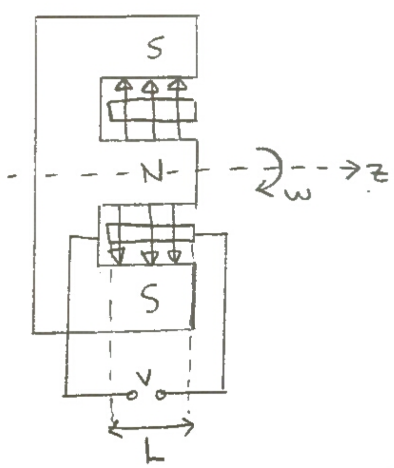
$$\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t} \xrightarrow[\text{integral}]{\text{surface}} \underbrace{\oint_c \mathbb{E} \cdot d\ell}_{v} = \underbrace{-\int_s \frac{\partial \mathbb{B}}{\partial t}\, dS}_{-\partial \phi / \partial t}$$

Stationary circuit in $\mathbb{B}(t)$
Moving ——— 〃 ——— $\longrightarrow v = -\partial \Phi / \partial t$

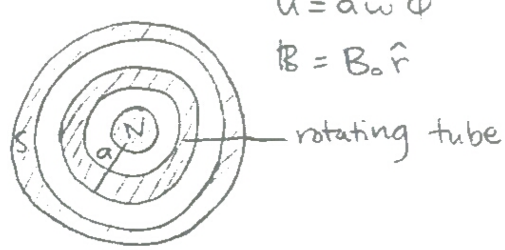
Moving conductor in a static magnetic field $\mathbb{B}_0$:

$$V_{21} = \int_1^2 (\underbrace{u}_{\text{circuit velocity}} \times \mathbb{B})\, d\ell \quad , \quad V = \oint_c (u \times \mathbb{B}) \cdot d\ell$$

10.4
A tube is rotating in a pemanent magnet.
$\Phi = 0{,}25$ wb , $V = 10$ V
How many $\underbrace{\text{rotations per minute}}_{N\,(\text{turns}/\text{minute})}$, tube has to give $V = 10$V?



$\hat{u} = a\omega\,\hat{\Phi}$
$\mathbb{B} = B_0 \hat{r}$



$$V_{12} = \int_1^2 (u \times \mathbb{B})\, d\ell = \int_1^2 (a\omega \hat{\Phi}) \times (B_0 \hat{r}) \cdot \hat{z}\, dz = \int_1^2 -\hat{z}\,(a\omega B_0)\,\hat{z}\, dz =$$

$$= -a\omega B_0 L$$

$$\Phi = \int_S B \cdot dS = 2\pi a L B_0$$ flux passing through cross sectional tube

$$V_{12} = -a\omega L B_0 = \frac{-\cancel{a}\omega \cancel{L}\Phi}{2\pi \cancel{a}\cancel{L}} = -\frac{\omega\Phi}{2\pi} \Longrightarrow \frac{\omega}{2\pi} = -\frac{V_{12}}{\Phi}$$

$$N = (\omega/2\pi) \cdot 60 = V_{12}/\Phi \cdot 60 = 10/0.25 \cdot 60 = 2400$$

Time harmonic $\mathbb{E}$-field: $\mathbb{E}(x,y,z,t) = \mathrm{Re}\{\underbrace{\bar{E}(x,y,z)}_{\text{vector phasor}} e^{j\omega t}\}$

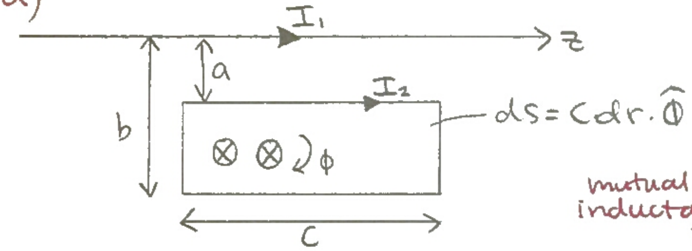
Magnetic force on a current carrying circuit:

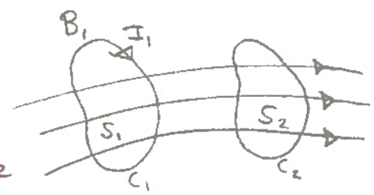
$$\mathbb{F}_m = I \oint_C d\ell \times \mathbb{B}$$ external magnetic field.

10.6

We have a rectangular loop with resistance R and inductance L located near a long wire with current I_1, $I_1 = I_0\cos(\omega t)$. Find the mutual inductance (L_{12})!

a)





mutual inductance

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_2 \Phi_{12}}{I_1}$$

flux linkage

$$\mathbb{B} = \frac{\mu_0 I_1}{2\pi r}\hat{\Phi}$$

$$\Phi_{12} = \int_S \mathbb{B}\cdot ds = \int_a^b \frac{\mu_0 I_1}{2\pi r}\hat{\Phi}\cdot c\,\hat{\Phi}\,dr =$$

$$= \frac{\mu_0 I_1 c}{2\pi}\left[\ln r\right]_a^b = \frac{\mu_0 I_1 c}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$L_{12} = \frac{\Phi_{12}}{I_1} = \frac{\mu_0 c}{2\pi}\ln\left(\frac{b}{a}\right)$$

b) Find current I_2!

$$V = \frac{-\partial\Phi}{\partial t} = \frac{-\partial}{\partial t}\left(\frac{\mu_0 \overbrace{I_1}^{I_0\cos(\omega t)} c}{2\pi}\ln\left(\frac{b}{a}\right)\right) = \frac{I_0\mu_0 c\omega}{2\pi}\ln\left(\frac{b}{a}\right)\underbrace{\sin(\omega t)}_{\cos(\omega t - \pi/2)}$$

$$V = \mathrm{Re}\{\bar{V}_1 e^{j(\omega t - \pi/2)}\} = \mathrm{Re}\{\underbrace{\bar{V}_1 e^{-j\pi/2}}_{\bar{V}_2} e^{j\omega t}\}$$

$$\Rightarrow \bar{V}_2 = \frac{I_0\mu_0 c\omega}{2\pi}\ln\left(\frac{b}{a}\right)e^{j\pi/2}$$

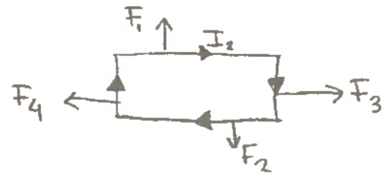
$$\bar{I}_2 = \frac{\bar{V}_2}{R + j\omega L} = \frac{\bar{V}_2(R - j\omega L)}{R^2 + \omega^2 L^2} = \frac{I_0\mu_0 c\omega\ln\left(\frac{b}{a}\right)}{2\pi(R^2+\omega^2L^2)}(R - j\omega L)e^{-j\pi/2}$$

forts. →

$$I_2 = \mathrm{Re}\{\bar{I}_2 \cdot e^{i\omega t}\} = \frac{I_0 \mu_0 c\, \omega \ln\left(\frac{b}{a}\right)}{2\pi(R^2+\omega^2 L^2)}\left[R\sin \omega t - \omega L\cos(\omega t)\right]$$

c) Calculate the magnetic force on the loop!

$$F_m = I \oint_c dl \times B$$



$$B = \frac{\mu_0 I_1}{2\pi r}\hat{\phi}$$

$$F_m = \frac{I_2 c \mu_0 I_1}{2\pi b}\hat{r} - \frac{I_2 c \mu_0 I_1}{2\pi a}\hat{r}$$

At short sides the forces cancel eachother.

$$F_m = \frac{\mu_0 c I_1 I_2}{2\pi}\left(\frac{1}{b} - \frac{1}{a}\right)\hat{r} =$$

$$= \hat{r}\,\frac{\mu_0 c}{2\pi}\left(\frac{1}{b} - \frac{1}{a}\right)\underbrace{I_0\cos(\omega t)}_{I_1}\,\underbrace{\frac{I_0 \mu_0 c\,\omega \ln\left(\frac{b}{a}\right)}{2\pi(R^2+\omega^2L^2)}\left[R\sin(\omega t) - \omega L\cos(\omega t)\right]}_{I_2}$$

d) Calculate time average force $\langle F_m \rangle$!

$$\langle F_m \rangle = \frac{1}{T}\int_0^T F_m(t)\,dt =$$

$$= \frac{1}{T}\hat{r}\,\underbrace{\left(\frac{\mu_0 c I_0}{2\pi}\right)^2\left(\frac{1}{b} - \frac{1}{a}\right)\frac{\omega \ln\left(\frac{b}{a}\right)}{R^2+\omega^2L^2}}_{\alpha}\int_0^T\left[R\sin(\omega t) - \omega L\cos^2(\omega t)\right]dt\left(\frac{\omega}{\omega}\right) =$$

$$T = \frac{2\pi}{\omega}$$

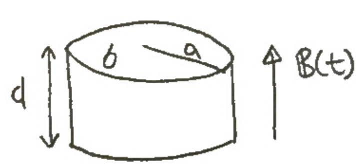
$$= \hat{r}\,\alpha\,\frac{\cancel{\omega}}{2\pi}\int_{\omega t=0}^{2\pi}\left[R\sin(\omega t) - \omega L\cos^2(\omega t)\right]\frac{d(\omega t)}{\cancel{\omega}} =$$

$$= \hat{r}\,\alpha\,\frac{1}{2\pi}\left[R\cdot 0 - \omega L\pi\right] = -\hat{r}\,\alpha\,\frac{\omega L}{2}$$

$$\left(\int_{\omega t=0}^{2\pi}\underbrace{\cos^2(\omega t)}_{\frac{1+\cos(2\omega t)}{2}}d(\omega t) = \left[\frac{\omega t}{2} + \frac{1}{4}\sin(2\omega t)\right]_0^{2\pi} = \pi\right)$$

12.22

A circular disk located in time varying uniform magn. field.



$$\begin{cases} B(t) = \hat{z}B_0\cos(\omega t) \\ \omega = 2\pi\cdot 10^3 \end{cases}$$

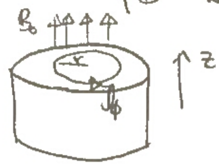
$$\begin{cases} a = 3\,cm \\ d = 0{,}1\,mm \\ \sigma = 10^7\,S/m \end{cases}$$

a) Calculate induced eddy currents!

$$\mathbb{J}_\phi(r) = \sigma \mathbb{E}_\phi(r)$$

$$\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t} \Longrightarrow \text{phasor form } \nabla \times \bar{\mathbb{E}} = -i\omega \bar{\mathbb{B}}$$

$$\xRightarrow{\text{surface integral}} \oint_c \mathbb{E} \cdot d\ell = -i\omega \int_s \mathbb{B}\, dS, \quad \begin{cases} \bar{\mathbb{E}} = \bar{\mathbb{E}}_\phi(r)\hat{\Phi} \\ \bar{\mathbb{B}} = \mathbb{B}\hat{z} = B_0 \hat{z} \end{cases}$$



$$\Longrightarrow \mathbb{E}_\phi(r) \cdot 2\pi r = -i\omega B_0 \pi r^2$$

$$\Longrightarrow \bar{\mathbb{E}}_\phi(r) = \frac{-i\omega B_0 r}{2}$$

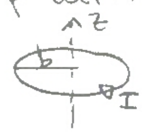
$$\bar{\mathbb{J}}_\phi(r) = \sigma \bar{\mathbb{E}}_\phi(r) = \frac{-i\sigma\omega B_0 r}{2}$$

$$\mathbb{J}_\phi(r,t) = \mathrm{Re}\{\mathbb{J}_\phi(r) e^{i\omega t}\} = \frac{\sigma B_0 \omega r}{2} \sin(\omega t)$$

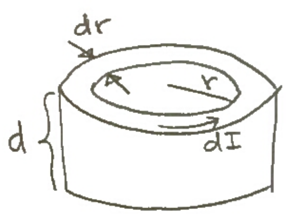
b) Calculate $B(0,t)$ caused by eddy current at center.
$d \ll a \Rightarrow$ neglect the thickness of the plate.
Consider the current as many current loops.

Magnetic field at the center of a circular loop with current I:

$$\mathbb{B} = \hat{z} \frac{\mu_0 I b^2}{2(\cancel{z^2} + b^2)^{3/2}} = \frac{\hat{z}\, \mu_0 I}{2b}$$



(page 288.)



$$d\mathbb{B} = \frac{\mu_0\, dI}{2r} \hat{z}$$

$$dI = \mathbb{J}_\phi \cdot d \cdot dr$$

$$\text{Total } \mathbb{B} = \int_{r=0}^{a} d\mathbb{B} = \int \frac{\mu_0 \mathbb{J}_\phi d}{2r} \hat{z}\, dr$$

$$\mathbb{B} = \int_0^a \frac{\mu_0 \sigma \omega B_0 r d}{4r} \sin(\omega t)\hat{z}\, dr = \frac{\mu_0 \sigma \omega B_0 d a}{4} \sin(\omega t)\hat{z}$$

$$\mathbb{B} = \frac{4\pi \cdot 10^{-7} \cdot 10^{7} \cdot 2\pi \cdot 10^{5} \cdot 0{,}1 \cdot 10^{-3} \cdot 3 \cdot 10^{-2} \cdot B_0 \sin(\omega t)}{4} \hat{z} =$$

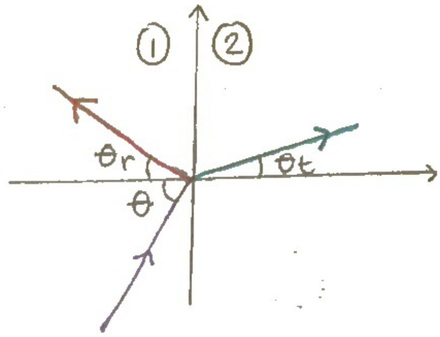
$$= 0{,}0592\, B_0 \sin(\omega t)\, \hat{z}$$

Föreläsning 4/12-13

Snells lag: $c_2 \sin\theta_i = c_1 \sin\theta_t$

Vad händer om $\varepsilon_1 > \varepsilon_2$ $\quad (c = 1/\sqrt{\varepsilon\mu}$

$c_1 < c_2$



Total reflektion 8.10.1

Antag $\mu_1 = \mu_2 = \mu_0$

Snells lag ger: $\sin\theta_t = \sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}} \sin\theta_i$

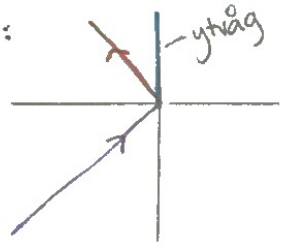
Fallet $\theta_t = \pi/2$

$\Rightarrow \theta_i = \arcsin\sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}} = \theta_{\text{kritiska vinkeln}}$

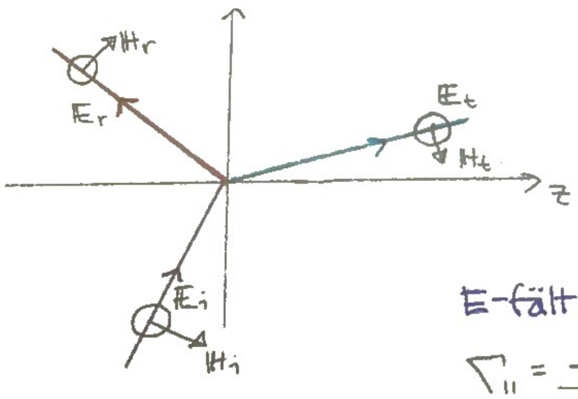
För $\theta_i > \theta_{\text{kritiska vinkeln}} = \theta_c$

$\Rightarrow \sin\theta_t \sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}} > 1$ ingen reell lösning

Vi får en ytvåg:



Fresnels ekvationer 8.10.2, 8.10.3



E-fält vinkelrätt mot infallsplanet:

$$\Gamma_\perp = \left(\frac{E_{r0}}{E_{i0}}\right)_\perp = \frac{1/z_1 \cos\theta_i - 1/z_2 \cos\theta_t}{1/z_1 \cos\theta_i + 1/z_2 \cos\theta_t}$$

$$T_\perp = \left(\frac{E_{t0}}{E_{i0}}\right)_\perp = \frac{2/z_1 \cos\theta_i}{1/z_1 \cos\theta_i + 1/z_2 \cos\theta_t}$$

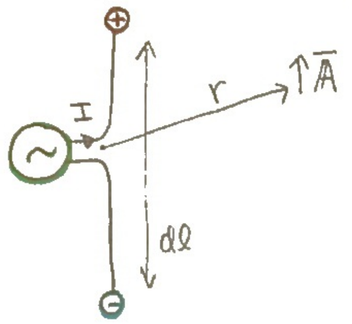
E-fält parallellt m. infallsplanet:

$$\Gamma_\parallel = \frac{-z_1 \cos\theta_i + z_2 \cos\theta_t}{z_1 \cos\theta_i + z_2 \cos\theta_t}$$

$$T_\parallel = \frac{2 z_2 \cos\theta_i}{z_1 \cos\theta_i + z_2 \cos\theta_t}$$

Brewstervinkeln:
kan hända att $Z_2\cos\theta_t = Z_1\cos\theta_i \Longrightarrow \Gamma_{\parallel} = 0.$

Antenner och Hertzdipolen 11.1, 11.2



Approx. $d\ell \ll \lambda$
$d\ell \ll r$

Låt I vara konstant längs $d\ell$ (*)

Retarderad potential

$$\mathbb{A} = \frac{\mu_0}{4\pi}\int_{V_1} \frac{\mathbb{J}(\mathbb{r}_1)\, e^{-j\frac{\omega r}{c}}}{r}\, dv_1$$

$$(*) \Longrightarrow \quad \bar{\mathbb{A}} = \frac{\mu_0}{4\pi}\frac{\bar{I}\, d\ell}{r} e^{-j\beta r}\,\hat{z}$$

$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

$$\begin{cases} A_r = A_z\cos\theta = \dfrac{\mu_0 I\, d\ell}{4\pi}\dfrac{e^{-j\beta r}}{r}\cos\theta \\ A_\theta = -A_z\sin\theta = -\dfrac{\mu_0 I\, d\ell}{4\pi}\dfrac{e^{-j\beta r}}{r}\sin\theta \\ A_\phi = 0 \end{cases}$$

$$\mathbb{H} = \frac{1}{\mu_0}\nabla\times\mathbb{A} = \hat{\Phi}\frac{1}{\mu_0 r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] = -\hat{\Phi}\frac{\bar{I}\, d\ell}{4\pi}\beta^2\sin\theta\left[\frac{1}{j\beta r} + \frac{1}{(j\beta r)^2}\right]e^{-j\beta r}$$

$$\mathbb{E} = \frac{1}{j\omega\varepsilon_0}\nabla\times\mathbb{H} = \frac{1}{j\omega\varepsilon_0}\left[\hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(H_\phi\sin\theta - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\right)\right]$$

$$\bar{E} = -\frac{\bar{I}\, d\ell}{4\pi}Z_0\beta^2\left[\hat{r}\left(\frac{2}{(j\beta r)^3} + \frac{2}{(j\beta r)^2}\right)\cos\theta + \hat{\theta}\left(\frac{1}{(j\beta r)^3} + \frac{1}{(j\beta r)^2} + \frac{1}{j\beta r}\right)\sin\theta\right]e^{-j\beta r}$$

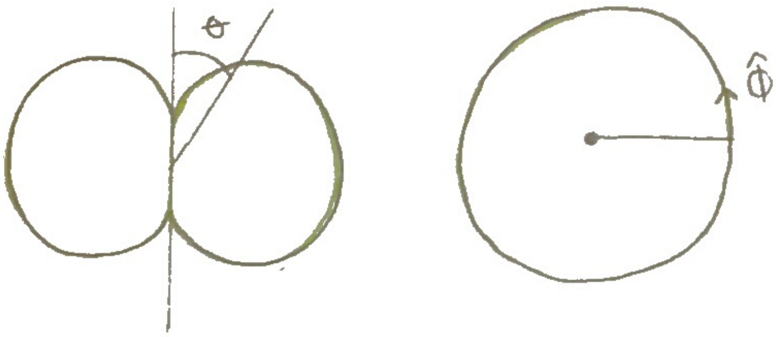
Approximera $r \gg \lambda$ (fjärrfält):

$r \gg 1/\beta = \lambda/2\pi \gg d\ell$

$$\bar{E}_\theta = i \frac{\bar{I}\,d\ell}{4\pi} \frac{e^{-i\beta r}}{r} Z_0 \beta \sin\theta$$

$$\bar{H}_\phi = i \frac{\bar{I}\,d\ell}{4\pi} \frac{e^{-i\beta r}}{r} \beta \sin\theta$$

Strålningsdiagram 11.3



θ-planet
E-plan

Φ-planet
H-plan

Strålningsresistans ex 11.3

$$S_{av} = \hat{r} \cdot \mathrm{Re}\, \tfrac{1}{2} \{ \bar{E} \times \bar{H}^* \}$$

$$P_{av} = \oint_S S_{av} \cdot dS = R_{rad} I_{eff}^2$$

yta

$$\bar{E} \times \bar{H}^* = \hat{\theta} Z_0 \frac{i\omega d\ell \bar{I} \sin\theta}{4\pi c r} e^{-i\beta r} \times \hat{\Phi} \frac{-i\omega d\ell \bar{I} \sin\theta}{4\pi c r} e^{i\beta r} =$$

$$= \hat{r} Z_0 \frac{\omega^2 d\ell^2 |\bar{I}|^2 \sin^2\theta}{16\pi^2 c^2 r^2}$$

$$\Rightarrow R_{rad} = 80\pi^2 \left(\frac{d\ell}{\lambda}\right)^2$$

Atomförstärkning 11.3

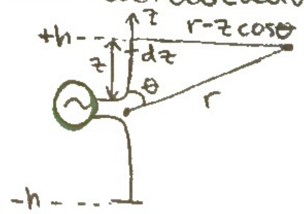
$$G_D(\theta, \phi) = \frac{S_{rad}(\theta,\phi)}{S_{isotrop}} = \frac{S_{rad}(\theta,\phi)}{P_{av}/4\pi r}$$

För Hertzdipol: $G_D(\theta,\phi) = 3/2 \cdot \sin^2\theta$

Direktivitet: $D = \max(G_D) \Longrightarrow D = 1{,}5$ för dipol

<u>Dipolantenner 11.4</u>

Med antennlängd $l \sim \lambda$



Hertzdipolbidrag:

$$d\mathbb{E}_{rad} = \hat{\theta}\, Z_0 \frac{i\omega\, dz\, \bar{I}(z) \sin\theta}{4\pi c r} e^{-i\beta(r - z\cos\theta)}$$

Antag $\bar{I}(z) = I_0 \sin\{\beta(h - |z|)\}$

$$\bar{E}_z = \int_{-h}^{h} d\mathbb{E}_{rad}$$

<u>Halvvågsantenn 11.4.1</u>

$$E_\theta = Z_0 H_\varphi = \ldots = i \frac{60 \bar{I}_0}{r} e^{-i\beta r} \left\{ \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \right\}$$

$$S_{av} = \mathrm{Re}\, \frac{1}{2}\left(E_{rad} + \mathbb{H}^*_{rad}\right) = \frac{15 I_0^2}{\pi r^2} \left(\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \right)^2$$

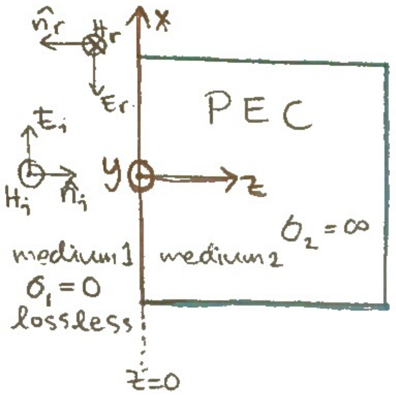
$$P_{av} = \int_S S_{av}\, dS = R_{rad} \frac{I_0^2}{2} = 36{,}5\, I_0^2$$

Strålningsresistans: $R_{rad} = 73{,}1\ \Omega$

Direktivitet: $D = 1{,}64$

Storgruppsövning 4/12-13

Normal incidence on conductor (plane wave)



$E_2 = H_2 = 0$ in medium 2

incident wave (E_i, H_i)

$E_i(z) = \hat{x} E_{i0} e^{-i\beta_1 z}$ phase constant of medium 1.

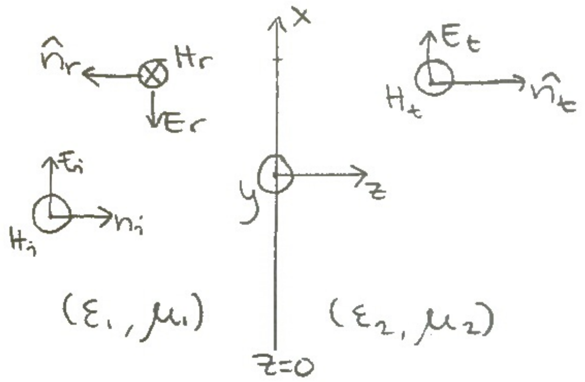
$H_i(z) = \frac{1}{\eta_1} \hat{n}_i \times E_i = \frac{E_{i0}}{\eta_1} \hat{y} e^{-i\beta_1 z}$

$E_1 = E_i + E_r = \hat{x} \left(E_{i0} e^{-i\beta_1 z} + E_{r0} e^{-i\beta_1 z}\right) = -\hat{x}\, 2i E_{i0} \sin\beta_1 z$

by writing the B.C at interface $E_1(0) = E_2(0) = 0 \Rightarrow (E_{r0} = -E_{i0})$

$$\begin{cases} H_r(z) = \frac{1}{\eta_1} \hat{n}_r \times E_r(z) \\ H_1 = H_i(z) + H_r(z) \end{cases}$$

Normal incidence on a dielectric boundary



$$\begin{cases} \bar{E}_i(z) = \hat{x} E_{i0} e^{-i\beta_1 z} \\ \bar{H}_i(z) = \hat{y} \frac{E_{i0}}{\eta_1} e^{-i\beta_1 z} \end{cases} \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

η_1 — intrinsic impedance

$$\begin{cases} \bar{E}_r(z) = \hat{x} E_{r0} e^{i\beta_1 z} \\ \bar{H}_r(z) = -\hat{y} \frac{E_{r0}}{\eta_1} e^{i\beta_1 z} \end{cases}$$

$$\begin{cases} \bar{E}_t(z) = \hat{x} E_{t0} e^{-\beta_2 z} \\ \bar{H}_t(z) = \hat{y} \frac{E_{t0}}{\eta_2} e^{-i\beta_2 z} \end{cases}$$

$\xRightarrow{B.C}$ $\bar{E}_i(0) + \bar{E}_r(0) = \bar{E}_t(0)$

$\bar{H}_i(0) + \bar{H}_r(0) = \bar{H}_t(0)$

$\Rightarrow \quad \Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \quad T = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_1 + \eta_2}, \quad 1 + \Gamma = T$

Plane wave in lossy media:

$\nabla^2 E - \gamma^2 E = 0$ source free wave equation

$\Rightarrow \bar{E} = \hat{x} E_0 e^{-\gamma z}$

$\gamma = \alpha + j\beta$

$\eta_c = Z_c = \sqrt{\frac{\mu}{\varepsilon_c}}$ intrinsic impedance

Good conductors:

$\sigma >> \omega\varepsilon$, $\gamma = \alpha + j\beta = (1+j)/\delta$ penetration coeff.

$Z_c = Z_0 \sqrt{\frac{\omega \mu_r \varepsilon_0}{2\sigma}}(1+j)$

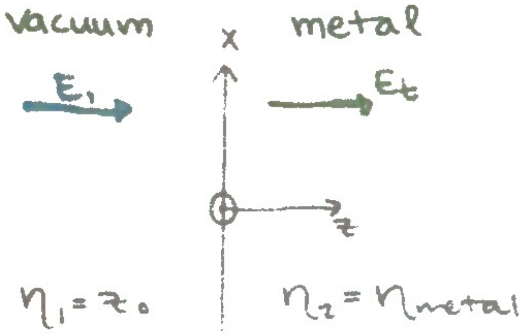
13.3

A plane wave in vacuum has normal incidence upon a conductive metal plate ($\delta = 1$mm)

– instantaneous E at boundary: ($E = \hat{x}\, 100\sqrt{2} \sin\omega t \cdot 10^{-6}$)

– %99 of the incident power is reflected ($|\Gamma|^2 = 0{,}99$)

Find the instantaneous E-field at z inside metal



$$\begin{cases} E_x(z=0^-, t) = 10^{-4}\sqrt{2}\, \overbrace{\sin\omega t}^{\cos(\omega t - \pi/2)} \\ \bar{E}_x(z=0^-) = 10^{-4}\sqrt{2}\,(-j) \end{cases}$$

$$\begin{cases} \text{in metal: } \bar{E}_x(z) = \tau \bar{E}_0 e^{-\gamma z} \quad ; \quad \gamma = (1+j)/\delta \\ \eta_2 = Z_c = Z_0 \underbrace{\sqrt{\frac{\omega \mu_r \varepsilon_0}{2\sigma}}}_{a}(1+j) = a(1+j)Z_0 \quad (a <<) \end{cases}$$

$$|\Gamma|^2 = 0{,}99 = \left|\frac{Z_c - Z_0}{Z_c + Z_0}\right|^2 = \left|\frac{a(1+j)-1}{a(1+j)+1}\right|^2 = \frac{(a-1)^2 + a^2}{(a+1)^2 + a^2} = \frac{2a^2 - 2a + 1}{2a^2 + 2a + 1}$$

$$\Rightarrow \begin{cases} a_1 = 0{,}0025 = 1/400 \\ a_2 = 198{,}99 \quad (a << 1) \end{cases}$$

$$\tau = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{2a(1+j)}{\underbrace{a\cancel{(1+j)}+1}_{a<<1}} \approx 2a(1+j) = 2\sqrt{2}\, a\, e^{j\pi/4}$$

$$\bar{E}_T = T\bar{E}_0 e^{-(1+i)\delta} = \frac{e^{i\pi/4}}{100\sqrt{2}} \left(10^{-4}\sqrt{2}(-i)\right) e^{-10^3 z} e^{-10^3 i z} =$$

$$= 10^{6} e^{-i\pi/4} e^{-10^3 z} e^{-i 10^3 z}$$

$$E_{Tx}(z,t) = Re\left\{\bar{E}_{Tx}(z) e^{i\omega t}\right\} = 10^{-6} e^{-10^3 z} \cos(\omega t - 10^3 z - \pi/4)$$

For a conducting media $(\delta \neq 0)$, $\mathbb{J} = \sigma \mathbb{E}$

$$\nabla \times \mathbb{H} = \mathbb{J} + i\omega \mathbb{D} = \sigma \mathbb{E} + i\omega\varepsilon\mathbb{E} = i\omega \underbrace{\left(\varepsilon + \frac{\sigma}{i\omega}\right)}_{\varepsilon_c \text{ - complex permittivity}} \mathbb{E}$$

$$\varepsilon_c = \varepsilon - i\frac{\sigma}{\omega}$$

1.3.5

Planear linearly polarized wave, $(\lambda = 30\text{cm})$, in vacuum normal incidence on water surface.
Calculate reflected power coef. : $|\Gamma|^2$ water : $\begin{cases} \sigma = 5 \text{ S/m} \\ \varepsilon_r = 80 \end{cases}$

$$\omega\varepsilon = \omega\varepsilon_0\varepsilon_r = 2\pi f \varepsilon_0 \varepsilon_r = 2\pi \cdot 10^9 \cdot \frac{1}{36\pi} \cdot 10^{-9} \cdot 80 = 4{,}45$$

$$f = \frac{c}{\lambda} = \frac{3\cdot 10^8}{30\cdot 10^{-2}} = 1\,GHz \qquad \Longrightarrow \quad \sigma \gg \omega\varepsilon \quad \sigma \ll \omega\varepsilon$$

$$Z_2 = \eta_2 = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon_0\left(\varepsilon_r - \frac{i\sigma}{\omega\varepsilon_0}\right)}} = \overbrace{\sqrt{\frac{\mu}{\varepsilon_0}}}^{Z_0} \sqrt{\frac{1}{\varepsilon_r - i\frac{\sigma}{\omega\varepsilon_0}}}$$

$$Z_2 = Z_0\sqrt{\frac{1}{8 - i90}}$$ intrinsic impedance of water ,

$$\begin{cases} \Gamma = \frac{Z_2 - Z_0}{Z_2 + Z_0} \Rightarrow |\Gamma| = 0{,}847 \\ |\Gamma|^2 = 0{,}717 \end{cases}$$

Poynting vector:
is a power density of an electromagnetic field.

$$\bar{\mathbb{P}} = \mathbb{E} \times \mathbb{H} \quad \left[\frac{W}{m^2}\right]$$

instantaneous power density : $\mathbb{P}(z,t) = \mathbb{E}(z,t) \times \mathbb{H}(z,t)$

average power density : $\mathbb{P}_{av} = \frac{1}{T}\int_0^T \mathbb{P}(z,t)\,dT$

$$\mathbb{P}_{av} = \frac{1}{2} Re\left\{\mathbb{E}(z) \times \mathbb{H}(z)^*\right\}$$

11.10

Linearly polarized plane wave, propagating through a lossless dielectric, $\varepsilon_r = 2.5$ $\mu_r = 1$, the wave has power density of 0.2 $[\frac{W}{m^2}]$. Find the peak values of E and H!

$$\begin{cases} \mathbb{E} = \mathbb{E}_0 e^{-ik\cdot R} \\ \mathbb{H} = \frac{1}{\eta}\hat{n}\times\mathbb{E} \end{cases}$$

$\begin{cases} k \text{ - wave number vector} \\ \hat{n} \text{ - direction of propagation} \end{cases}$

$$\begin{cases} \bar{\mathbb{E}} = \bar{\mathbb{E}}_0 e^{-ikz}\hat{x} \\ \mathbb{H} = \frac{\bar{\mathbb{E}}_0}{\eta} e^{-ikz}\hat{y} \end{cases}$$

we assume E_x, H_y travelling in z direction.

$$P_{av} = \frac{1}{2}\mathrm{Re}\{\mathbb{E}\times\mathbb{H}^*\} = \frac{1}{2}\mathrm{Re}\left\{(\mathbb{E}_0 e^{-ikz}\hat{x})\times\left(\frac{\bar{\mathbb{E}}_0^*}{\eta}e^{-ikz}\hat{y}\right)\right\} =$$

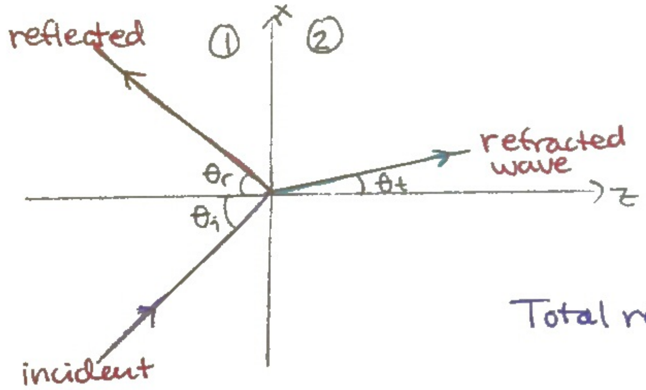
$$= \frac{1}{2\eta}|\bar{\mathbb{E}}_0|^2\hat{z} \Longrightarrow |\mathbb{E}_0|^2 = P_{av}\cdot 2\eta = 0{,}2\cdot 2\eta$$

$$\eta = z_c = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = z_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = 120\pi\sqrt{\frac{1}{2{,}5}} \approx 238{,}43$$

$$|\mathbb{E}_0| = \sqrt{0{,}4\cdot 238{,}43} \approx 9{,}77$$

$$|\mathbb{H}_0| = \frac{|\mathbb{E}_0|}{\eta} \approx 0{,}041$$

Oblique incidence at a plane dielectric boundary



Snell's law of refraction:

$\theta_i = \theta_r$

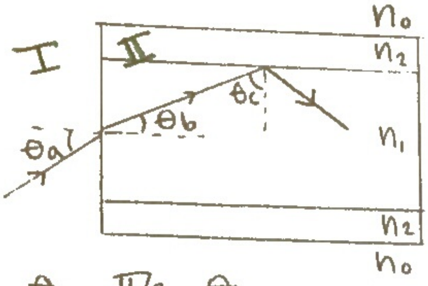
$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} = \frac{\beta_1}{\beta_2} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}$$

$$n = \frac{c}{u_p} = \sqrt{\varepsilon_r\mu_r}$$

Total reflection: $\theta_i = \theta_c \Rightarrow \theta_t = \pi/2$

P 8.41

Find the maximum θ_a so that the ray will be trapped inside.



We need total reflection at II

(I) $n_0 \sin\theta_a = n_1 \sin\theta_b$ refraction

(II) $n_1 \sin\theta_c = n_2 \sin(\pi/2)$ total reflection

$\theta_c = \pi/2 - \theta_b$

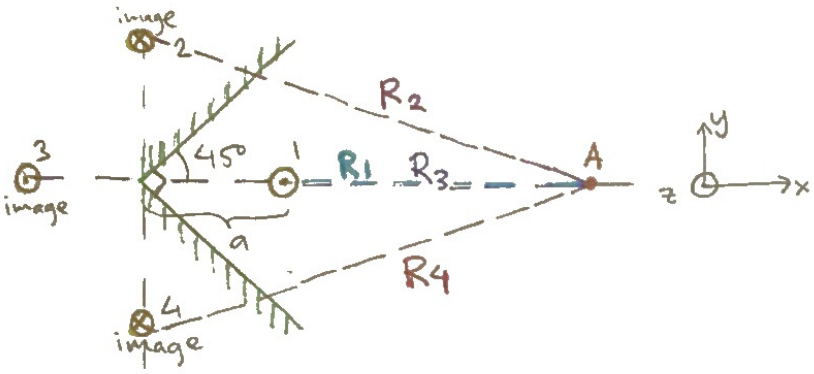
$n_1 \underbrace{\sin(\frac{\pi}{2} - \theta_b)}_{\cos\theta_b} = n_2 \Rightarrow \cos\theta_b = \frac{n_2}{n_1} \Rightarrow \underbrace{\sin\theta_b = \sqrt{1-\left(\frac{n_2}{n_1}\right)^2}}_{\text{substitute in I}}$

$\Rightarrow n_0 \sin\theta_a = n_1 \sqrt{1 - \left(\frac{n_2}{n_1}\right)^2} = \sqrt{n_1^2 - n_2^2}$

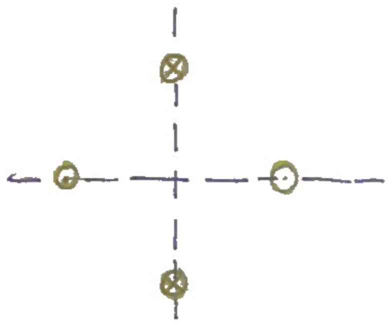
$\Rightarrow \sin\theta_a = \frac{\sqrt{n_1^2 - n_2^2}}{n_0} \Rightarrow \theta_a = \arcsin\left(\frac{\sqrt{n_1^2 - n_2^2}}{n_0}\right)$, $\theta_i \le \theta_a$

Storgruppsövning 6/12-13

16.1
A dipole is in front of two very long, perpendicular conductive plate. Field is observed at very long distance at $\theta = \pi/2$.
Find distance {a} such that the E-field in point {A} is maximised.



hint: use image method.



for field for each dipole antenna is:

$$\bar{E} = \bar{E}_0 F(\theta, \phi)\, e^{-i\beta R}/R$$

$$\bar{E}_\theta = i \frac{I d\ell}{4\pi} \left(\frac{e^{-i\beta R}}{R} \right) \eta \beta \sin\theta$$

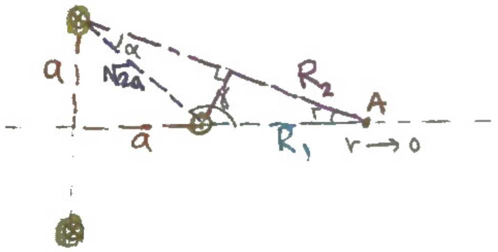
total field at point A, is the summation of the fields of 4 dipoles.

$F(\theta, \phi)$ and $1/R$ can be approximated the same for all antennas, but for the phase term we need more accurate approximation.

$$\bar{E}_{tot} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 + \bar{E}_4 = \frac{\bar{E}}{R_0} F(\theta,\phi)\, e^{-i\beta R_1} - \frac{\bar{E}}{R_0} F(\theta,\phi)\, e^{-i\beta R_2} +$$

$$+ \frac{\bar{E}}{R_0} F(\theta,\phi)\, e^{-i\beta R_3} - \frac{\bar{E}}{R_0} F(\theta,\phi)\, e^{-i\beta R_4}$$

$$\bar{E}_{tot} = \frac{\bar{E}}{R_0} F(\theta,\phi) \left[e^{-i\beta R_1} - e^{-i\beta R_2} + e^{-i\beta R_3} - e^{-i\beta R_4} \right]$$



$R_4 \approx R_1 + a\sqrt{2}\cos\alpha \quad (\alpha \approx 45°)$
$R_4 \approx R_1 + a$

Use cosine line in triangel:

$$R_4^2 = R_1^2 + (a\sqrt{2})^2 - 2R_1 a\sqrt{2}\cos(\gamma) = R_1^2 + 2aR_1 + 2a^2 =$$
$$= (R_1+a)^2 + a^2$$

$$\begin{cases} R_1 \gg a \implies R_4^2 \approx (R_1+a)^2 \implies R_4 \approx R_1 + a \\ R_4 = R_2 \\ R_3 = 2a + R_1 \end{cases}$$

$$\bar{E}_{tot} = \frac{\bar{E}_0}{R_0} F(\theta,\phi)\left[e^{-j\beta R_0} - e^{-j\beta(R_0+a)} + e^{-j\beta(R_0+2a)} - e^{-j\beta(R_0+a)}\right] =$$

$$= \frac{\bar{E}_0}{R_0} F(\theta,\phi)\left[e^{-j\beta R_0}\left(1 - 2e^{-j\beta a} + e^{-2j\beta a}\right)\right] =$$

$$= \frac{\bar{E}_0}{R_0} F(\theta,\phi)\left[e^{-j\beta R_0}\left(1 - e^{-j\beta a}\right)^2\right]$$

$$\implies \left|\bar{E}_{tot}\right| = \frac{|\bar{E}_0|}{R_0}\,|F(\theta,\phi)|\,\left|(1-e^{-j\beta a})\right|^2$$

$\bar{E}_{tot}$ is maximised when $|(1-e^{-j\beta a})|^2$ is maximised.

$$|(1-e^{-j\beta a})|^2 = |1-\cos\beta a + j\sin\beta a|^2 = (1-\cos\beta a)^2 + \sin^2\beta a =$$
$$= 2 - 2\cos\beta a = 2(1-\cos\beta a)$$

when $\cos\beta a = -1$, $|\bar{E}_{tot}|$ has the max value.

$$\cos\beta a = -1 \implies \beta a = (2n+1)\pi \implies a = (2n+1)\pi/\beta$$

$$\beta = \frac{2\pi}{\lambda} \implies a = \left(n+\tfrac{1}{2}\right)\lambda \qquad n = 0, 1, 2 \ldots$$

$$|\bar{E}_{tot}| = \frac{4|E_0|}{R} F(\theta,\phi) \quad \text{for } \cos\beta a = -1$$

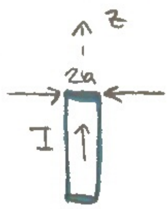
P 11.5 a)

Very thin center- half-ware dipole lying along z-axis.

Current dist. : $\begin{cases} I = I_0\cos\beta z \\ \beta = \frac{\omega}{c} = \frac{2\pi}{\lambda} \end{cases}$

$l = \frac{\lambda}{2}$

$S = \pi a^2$



Find the charge distribution on dipole.

$\nabla \cdot \bar{J} + j\omega\rho = 0$ continuity equation

Only z depending for both current and charge.

$$I = J_z(z)\cdot\pi a^2 = I_0\cos\beta z \implies J_z(z) = \frac{I_0\cos\beta z}{\pi a^2} \quad \left[\frac{A}{m^2}\right]$$

$$\nabla \cdot J_z(z) = \frac{\partial J_z}{\partial z} = -i\omega \rho \Longrightarrow \rho = \frac{-1}{i\omega} \frac{\partial J_z}{\partial z}$$

$\rho_\ell = \pi a^2 \cdot \rho =$

volume charge dist.

line charge dist.

$$= \pi a^2 \cdot \frac{-1}{i\omega} \frac{\partial J_z}{\partial z} = \pi a^2 \cdot \frac{-1}{i\omega} \cdot \frac{\partial \left(\frac{I_0 \cos \beta z}{\pi a^2} \right)}{\partial z} =$$

$$= \frac{-1}{i\omega} \frac{\partial (I_0 \cos \beta z)}{\partial z} = \frac{-1}{i\omega} (-I_0 \beta \sin \beta z) = \frac{\beta}{i\omega} I_0 \sin \beta z = \left\{ \beta = \frac{\omega}{c} \right\} =$$

$$= \frac{(\omega/c)}{i\omega} I_0 \sin \beta z = -i \frac{I_0}{c} \sin \beta z$$

Föreläsning 10/12-13

Elektrodynamik - repetition

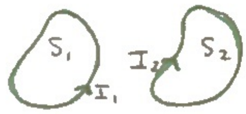
Postulaten:

Reell	Komplex
$\nabla\times\mathbb{E} = -\frac{\partial\mathbb{B}}{\partial t}$	$\nabla\times\bar{\mathbb{E}} = -i\omega\bar{\mathbb{B}}$
$\nabla\times\mathbb{H} = \mathbb{J} + \frac{\partial\mathbb{D}}{\partial t}$	$\nabla\times\bar{\mathbb{H}} = \bar{\mathbb{J}} + i\omega\bar{\mathbb{D}}$
$\nabla\cdot\mathbb{D} = \rho$	$\nabla\cdot\bar{\mathbb{D}} = \rho$
$\nabla\cdot\mathbb{B} = 0$	$\nabla\cdot\bar{\mathbb{B}} = 0$

Kont. ekv.: $\nabla\cdot\mathbb{J} + \frac{\partial\rho}{\partial t} = 0$

$\mathbb{F} = q(\mathbb{E} + \mathbb{u}\times\mathbb{B})$ Brentzkraft

Självinduktans: $\Phi = LI$, $\mathcal{E}_{ind} = -L\frac{\partial I}{\partial t}$



Ömsesidig induktans: $\Phi_{12} = \int_{S_2} \mathbb{B}_1 \cdot d\mathbb{S}_2 = L_{12} i_1$

Beräkningsgång

1. Antag I_1
2. Beräkna $\mathbb{B}_1$
3. Beräkna Φ_{12}
4. Beräkna Λ_{12}
5. Induktansen Λ_{12}/I_1

Induktion: $V_{ind} = -\frac{\partial\Phi}{\partial t}$

Lentz lag: inducerade spänningar motverkar förändringar i pålagt fält.

Retarderade potentialer: $\mathbb{A}(\mathbb{r}_2, t) = \frac{\mu_0}{4\pi}\int_{V_1} \frac{\mathbb{J}(\mathbb{r}_1, t - r_{12}/c)}{r_{12}}\, dV'$

$$V(\mathbb{r}_2, t) = \frac{1}{4\pi\varepsilon_0}\int_{V_1} \frac{\rho(\mathbb{r}_1, t - r_{12}/c)}{r_{12}}\, dV'$$

Komplexa fält: $\mathbb{E}(\mathbb{r}, t) = \mathrm{Re}\{\bar{\mathbb{E}}(\mathbb{r})e^{i\omega t}$

Komplexa vågekv.: $\nabla^2\bar{\mathbb{E}} - \gamma^2\mathbb{E} = 0$, $\gamma = \alpha + i\beta = \sqrt{i\omega\mu(i\omega\varepsilon + \sigma)}$

Specialfall: $\sigma/\varepsilon\omega \gg 1$, $\sigma/\varepsilon\omega \ll 1$

Plan våg: $\bar{\mathbb{E}}(\mathbb{R}) = \bar{\mathbb{E}}(0)e^{-\gamma \hat{k}\mathbb{R}}$, $\hat{\mathbb{k}} \cdot \bar{\mathbb{E}} = 0$ $(\mathbb{E} \perp \hat{\mathbb{k}})$

$\mathrm{Re}\{\mathbb{E} \cdot \bar{\mathbb{H}}^*\} = 0$

Vågimpedans: $Z = i\omega\mu/\gamma$

Relation mellan $\mathbb{E}$ och $\mathbb{H}$: $\bar{\mathbb{H}}(\mathbb{R}) = \hat{\mathbb{k}} \times \frac{\mathbb{E}(\mathbb{R})}{Z}$

Fashastighet: $V_{fas} = \omega/\beta$

Grupphastighet: $V_{grupp} = 1/\frac{\partial\beta}{\partial\omega}$

Poyntingvektorn: $\mathbb{S} = \mathbb{E} \times \mathbb{H}$

Tidsmv.: $\mathbb{S}_{av} = \frac{1}{2}\mathrm{Re}\{\bar{\mathbb{E}} \times \bar{\mathbb{H}}\}$

Reflektion och transmission:

Vinkelrätt infall: $\Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1}$, $\tau = \frac{2Z_2}{Z_2 + Z_1}$

Snells lag: $\theta_i = \theta_r$, $C_2 \sin\theta_i = C_1 \sin\theta_t$

Totalreflektion: $\theta_{kritisk} = \arcsin(C_1/C_2)$

Fresnels ekv.: $$\Gamma_\perp = \frac{(1/Z_1)\cdot\cos\theta_i - (1/Z_2)\cdot\cos\theta_t}{(1/Z_1)\cos\theta_i + (1/Z_2)\cos\theta_t}$$

$$\tau_\perp = \frac{(2/Z_1)\cos\theta_i}{(1/Z_1)\cos\theta_i + (1/Z_2)\cos\theta_t}$$

$$\Gamma_{\parallel} = \frac{-Z_1\cos\theta_i + Z_2\cos\theta_t}{Z_1\cos\theta_i + Z_2\cos\theta_t}$$

$$\tau_{\parallel} = \frac{2Z_2\cos\theta_i}{Z_1\cos\theta_i + Z_2\cos\theta_t}$$

Hertzdipol: $$\bar{\mathbb{E}}_{rad} = \frac{\hat{\theta} Z_0 i\omega\, dl\, \bar{I}\sin\theta}{4\pi c r} e^{-i\beta r}$$

$$\bar{\mathbb{H}}_{rad} = \frac{\hat{\varphi}\, i\omega\, dl\, \bar{I}\sin\theta}{4\pi c r} e^{-i\beta r}$$

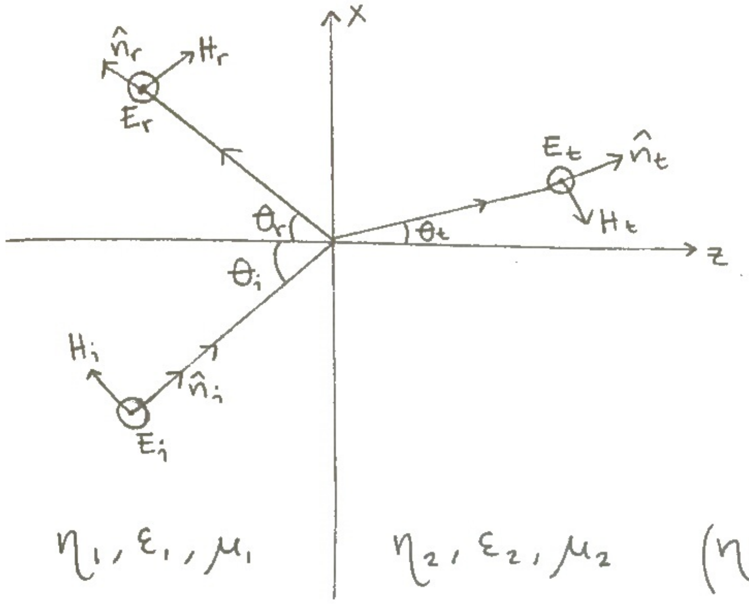
Dipolantenn: $E_z = \int_{-h}^{h} d\mathbb{E}_{rad}$

Storgruppsövning 10/12-13

Oblique incidence at a plane dielectric boundary.

Perpendicular polarization

E-field is perpendicular to the plane of incidence.



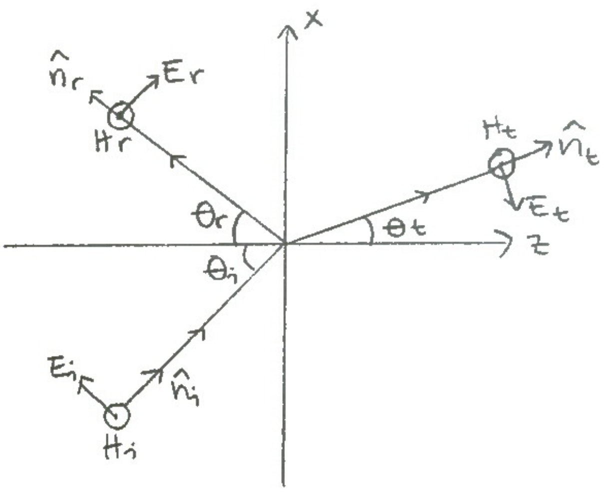
$$\left(\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}\right)$$

$$\begin{cases} \Gamma_\perp = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \\ \tau_\perp = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \end{cases}$$

$$1 + \Gamma_\perp = \tau_\perp$$

Parallell polarization

E-field is lying in the plane of incidence.



$$\begin{cases} \Gamma_{\parallel} = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \\ \tau_{\parallel} = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \end{cases}$$

$$1 + \Gamma_{\parallel} = \tau_{\parallel}$$

No reflection $\Longleftrightarrow$ θ_i = Brewster angle = θ_B

$\Gamma_\perp = 0 \Longrightarrow \eta_2 \cos\theta_i = \eta_2 \cos\theta_t$

$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} \xrightarrow[\text{law}]{\text{Snell's}} \cos\theta_t = \sqrt{1 - \left(\frac{\eta_1}{\eta_2}\right)^2 \sin^2\theta_t}$

$\Longrightarrow \sin\theta_i = \sin\theta_{B\perp} = \dfrac{1 - \mu_1\varepsilon_2/\mu_2\varepsilon_1}{1 - \left(\frac{\mu_1}{\mu_2}\right)^2}$

if $\mu_1 = \mu_2 \Longrightarrow \theta_B$ does not exist.

$\Gamma_{\parallel} = 0 \Longrightarrow \begin{cases} \eta_2 \cos\theta_t = \eta_1 \cos\theta_i \\ \cos\theta_t = \sqrt{1 - \sin^2\theta_t} \end{cases} \Longrightarrow \sin\theta_i = \sin\theta_{B\parallel} =$

$= \dfrac{1 - \mu_2\varepsilon_1/\mu_1\varepsilon_2}{1 - \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2}$

$\mu_2 = \mu_1 \Longrightarrow \sin\theta_{B\parallel} = \dfrac{1}{\sqrt{1 + \frac{\varepsilon_1}{\varepsilon_2}}} \Longrightarrow \tan\theta_{B\parallel} = \dfrac{\eta_2}{\eta_1}$

12.12

Circular cross section wire.
Radius $a = 0{,}1\,mm$
$\sigma = 5 \cdot 10^6\, S/m$
$\mu_r = 100$.
Calculate the ratio of the resistance at $f_1 = 50\,Hz$ & $f_2 = 10\,MHz$

Skin depth for good conductor, $\delta = \sqrt{\dfrac{1}{\pi f \mu \sigma}}$ $\left(\sigma/\omega\varepsilon \gg 1\right)$

$\begin{cases} f_1 = 50\,Hz \Longrightarrow \delta_1 = 3{,}2 \cdot 10^{-3} \\ f_2 = 10 \cdot 10^6 \Longrightarrow \delta_2 = 7 \cdot 11 \cdot 10^{-6} \end{cases}$ $(a = 0{,}1\,mm)$

$\begin{cases} \delta_1 \gg a \Longrightarrow \text{current dist. on the whole cross-section.} \\ \delta_2 \ll a \Longrightarrow \text{— '' — on a thin layer} \end{cases}$

$R_1 = \dfrac{l}{\sigma S_1} = \dfrac{l}{\sigma \pi a^2}$

$R_2 = \dfrac{l}{\sigma S_2} = \dfrac{l}{\sigma 2\pi a \delta_2}$

$\dfrac{R_1}{R_2} = \dfrac{S_2}{S_1} = \dfrac{2\delta_2}{a} \simeq 0{,}142\ldots$

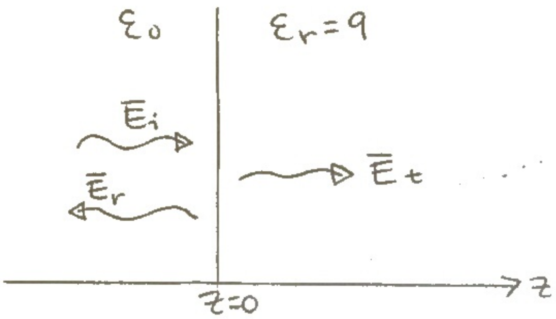
The resistance at 50Hz is approx 14% of that for 10 MHz.

13.7

A plane wave in vacuum incident normally to a flat surface at $z=0$, of a lossless dielectric, $\varepsilon_r = 9$.

$\mathbb{E} = \hat{x}\,10\cos(\omega t - \beta z)$, $f = 300\,\mathrm{MHz}$

Find the $\underbrace{\text{location}}_{z}$ of $\underbrace{\text{max for E-field in vacuum.}}_{\max(E_{tot}),}$



$$\bar{\mathbb{E}}_i = \hat{x}\,10e^{-i\beta z} \quad , \quad \bar{\mathbb{E}}_r = \Gamma \bar{\mathbb{E}}_i = \Gamma \hat{x}\,10e^{+i\beta z}$$

$$\bar{\mathbb{E}}_{tot} = \bar{\mathbb{E}}_i + \bar{\mathbb{E}}_r$$

$$\Gamma = \text{refl. coef.} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon_r}} - \sqrt{\frac{\mu_0}{\varepsilon_0}}}{\sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon_r}} + \sqrt{\frac{\mu_0}{\varepsilon_0}}} = \frac{1/\sqrt{\varepsilon_r} - 1}{1/\sqrt{\varepsilon_r} + 1} = \frac{1/3 - 1}{1/3 + 1} = -\frac{1}{2}$$

$$\Longrightarrow \bar{E}_r = -5\hat{x}e^{+i\beta z}$$

$$\bar{E}_{tot,1} = \hat{x}(10e^{-i\beta z} - 5e^{i\beta z}) = \hat{x}(10 - 5e^{2i\beta z})e^{-i\beta z}$$

$$\Longrightarrow |\bar{\mathbb{E}}_{tot,1}| = |10 - 5e^{2i\beta z}| = |\underbrace{10 - 5\cos(2\beta z)}_{Re} - \underbrace{5i\sin(2\beta z)}_{Im}|$$

$$|\bar{\mathbb{E}}_{tot,1}| = \sqrt{(10 - 5\cos(2\beta z))^2 + 25\sin^2(2\beta z)} = \sqrt{125 - 100\underbrace{\cos(2\beta z)}_{-1 \Rightarrow \max}}$$

$$\cos(2\beta z) = -1 \Longrightarrow |\bar{E}_{tot,1}|_{max} = \sqrt{225} = 15$$

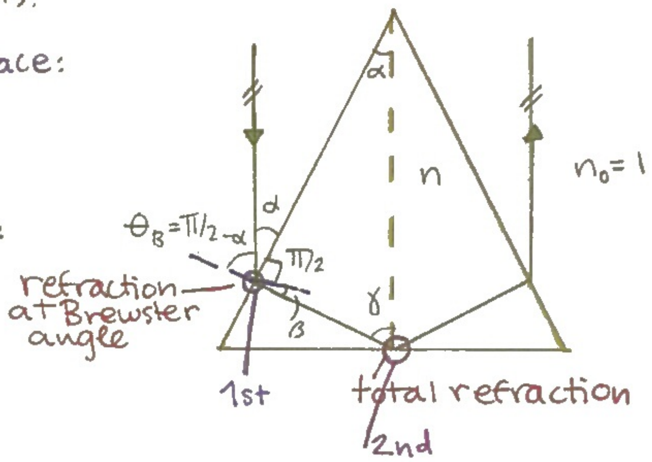
$$2\beta z_{max} = -(2n+1)\pi \Longrightarrow z_{max} = \frac{-(2n+1)\pi}{2\beta}$$

$$\beta = \frac{\omega}{c} = \frac{2\pi \cdot 3 \cdot 10^8}{3 \cdot 10^8} = 2\pi \Longrightarrow z_{max} = \frac{-(2n+1)\cancel{\pi}}{2 \cdot 2\cancel{\pi}} = \frac{-(2n+1)}{4}$$

$(n = 0, 1, 2, \dots)$

13.14

A light beam is broken and totally reflected in a lossless prism. Refraction is at Brewster angle. Return wave is parallell to incident wave.
Find a suitable range for (n).



① Brewster angle at 1st interface:

$$\tan\theta_B = \frac{n}{n_0} = n$$

$$\tan(\pi/2-\alpha) = n$$

② Snell's law of refr., 1st interf.:

$$\sin(\pi/2-\alpha) = n\sin\beta$$

③ Total reflection, 2nd interf.:

$$\gamma > \theta_c$$

④ Snell's law, 2nd interf.:

$$\sin\theta_c \cdot n = \sin\pi/2 \cdot 1$$

$$\sin\theta_c = 1/n$$

⑤ In triangle:

$$\alpha + (\pi/2 + \beta) + \gamma = \pi$$

Suppose $\gamma = \theta_c$ ($\tan\alpha = 1/n$)

① $\sin(\pi/2-\alpha) = n\cos(\pi/2-\alpha) \Longrightarrow \cos\alpha = n\sin\alpha$

② $\cos\alpha = n\sin\beta$

$\Longrightarrow \alpha = \beta$

④, ⑤ : $\sin\gamma = \sin\theta_c = \sin(\pi/2 - (\alpha+\beta)) = 1/n$

$$\Longrightarrow \cos(\alpha+\beta) = 1/n \Longrightarrow \cos 2\alpha = 1/n$$

$$\cos(2\alpha) = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = 1/n \Longrightarrow \{\tan(\alpha) = 1/n\} \Longrightarrow \frac{1-(1/n)^2}{1+(1/n)^2} = 1/n$$

$$\Longrightarrow n^3 - n^2 - n - 1 = 0$$

$$\Longrightarrow n_c \approx 1{,}839$$

If n increase $\Longrightarrow$ from ① $\alpha = \beta$ will decrease $\Longrightarrow \gamma$ will decrease
from ④ θ_c will decrease

So if $n \geq n_c \Longrightarrow \gamma \geq \theta_c \Longrightarrow$ total refraction at 2nd interface